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École doctorale

« R&D collaborations and innovation clusters: a theoretical approach »

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Mémoire présenté pour le Master en
ÉCONOMIE
EPP, PhD Track

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Paris, le 20 mai 2014
Année 2013/2014
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Abstract

We examine a model of innovation oligopoly in which technological spillovers are determined both by explicit links of collaboration between firms and by the choice of locating inside or outside an industrial cluster. The structure of the collaborations network is exogenous while we endogenize location choice. We show that the level of output (R&D effort and quantities) is proportional to the centrality in the network in this framework. Moreover, we show that the cluster will always host the most central firms in the network. We address the effect of changing the locational equilibrium on aggregate profits and welfare. We also examine the stability of the network ex post. Finally, we assess the effect of two instruments of public policy, and we simulate the effect of strengthening the level of cluster spillovers in a star network with three firms.

Paris, May 19, 2014
# Contents

1 Literature review ........................ 4  
   1.1 Localization of spillovers and industrial clusters .............................. 4  
   1.2 R&D collaboration networks ......................................................... 5  
   1.3 Theoretical literature on social networks ...................................... 5  

2 The model ................................. 6  
   2.1 Goals ................................................. 6  
   2.2 Tools for network analysis ......................................................... 7  
   2.3 The environment ......................................................... 8  

3 Equilibrium with exogenous location ................................................. 10  

4 Introducing location choice ................................................. 13  
   4.1 Incentives to move into the cluster ............................................... 13  
   4.2 Equilibria ................................................. 16  

5 Comparative statics and welfare analysis ............................................. 17  
   5.1 Effect of an entry on aggregate quantities ................................... 18  
   5.2 Effect of an entry on aggregate profits ....................................... 18  
   5.3 Effect of an entry on aggregate utility ....................................... 21  
   5.4 Effect on profits of non-entrants ............................................... 22  

6 Stability of the network ................................................. 23  
   6.1 Incentives to create new links ................................................... 24  
   6.2 Incentives to sever links ......................................................... 28  

7 Public policy ................................................. 29  
   7.1 Increase in the intensity of cluster spillovers ................................ 29  
   7.2 Linear R&D effort subsidy ......................................................... 31  
   7.3 Insights on the key-player policy ............................................... 32  

8 Simulations of outcomes in a star network with 3 firms ........................ 33  
   8.1 Effects of locations and intensity of cluster spillovers on quantities and profits ................................................. 33  
   8.1.1 Profits under high competition and low collaboration spillovers ...... 33  
   8.1.2 Profits with high collaboration spillovers .................................... 36  
   8.2 Welfare analysis in the 3 firms star network .................................. 37  

9 Conclusion ................................................. 39  

Appendices ................................................. 41
Introduction

Knowledge spillovers relate to the fact that efforts in R&D of an actor in an innovative market end up facilitating other actors’ efforts. They have become a major field of studies since the end of the 1980s, through the endogenous growth theory that considered technological externalities as one of the main sources of growth (Arrow (1962), Romer (1986)), and therefore made of this phenomenon a considerable issue in economic research. These macroeconomic models were actually a rediscovery of Marshallian agglomeration externalities, as exposed in Marshall (1890). Thus, the very idea of spillovers is linked with localization because they rank among agglomeration forces in MAR (Marshall-Arrow-Romer) externalities.

In the microeconomic literature, these spillovers have been integrated to an oligopoly competition model by the seminal paper by d’Aspremont and Jacquemin (1988). In their framework, knowledge flows out costlessly at a certain rate between the competitors. It was the first paper introducing explicitly ambiguous behaviors of competition and cooperation among firms in an innovative market. However, the precise process that fosters this sharing is not extensively covered in their model, that does not modelize explicitly a way through which information circulates, may it be a local link or an explicit collaborative link.

Later on, the social networks literature has focused on R&D collaborations, by extending the framework of d’Aspremont and Jacquemin (1988) to a framework by Goyal and Moraga-Gonzalez (2001) were only firms that clinched an explicit partnership received a share of their partner’s efforts. We build on this branch of R&D networks literature in our model.

We then want to propose a theoretical framework for knowledge spillovers using two major streams in the recent research:

- spatial neighborhood and localization of spillovers;
- technological neighborhood, via explicit R&D cooperation networks.

The ultimate goal of such a model is to provide insights for policy making, by showing various effects of the introduction of a cluster in an innovative market where some firms collaborate. Indeed, public support to innovation has been promoted in Europe since 2002 and the Barcelona objectives (Czarnitzki and Fier (2003)), and has taken various forms, such as direct R&D subsidies, public initiative for the creation of clusters, etc. One may want to address the effect of this public support on aggregate R&D efforts, on welfare and on firms profits. In particular, the opportunity of promoting clusters may depend a lot on the industry level of spillovers, and on the preexisting network of collaboration between firms. This may then involve targeted support, following the path opened by Ballester et al. (2006)

I thank my adviser Emeric Henry, for very helpful comments, and Yves Zenou for providing me precious advice on recent literature that turned out to be the basis of this master thesis.

there are three types of agglomeration externalities in the marshallian analysis: sharing of facilities, matching externalities in the labor market, and learning and knowledge externalities, on which we focus here.
on key-player policies.

In a first section, we review the literature on the three branches on which we build, namely the localization of spillovers and the cluster policies, the R&D collaborations, and the theoretical literature on social networks. In section 2, we describe the environment of our model, and then solve it with exogenous locations (i.e. the second stage of our game) in section 3, and endogenize the location choice and solve the first stage of the game in section 4. In section 5, we provide some comparative statics when we change the first stage equilibrium. In section 6, we give some insights on the stability of the collaborations network in equilibrium. Finally, in section 7, we analyse tools for public policies, and we simulate the use of one of these tools in a simple network in section 8.

1 Literature review

1.1 Localization of spillovers and industrial clusters

Since Jaffe (1989) and Jaffe et al. (1993), many papers have shown empirically that knowledge spillovers diffused locally. These articles focused on different scales: some papers focused on the metropolitan level (see Anselin et al. (1997)), on the regional level (Bottazzi and Peri (2003) and Greunz (2003) on European data, Jaffe et al. (1993) on American data), and on the international level (see Keller (2001)). Research also focused on different types of actors: inter firms spillovers in Jaffe et al. (1993), effects of academic research on firms spillovers in Anselin et al. (1997), and effect of the mobility of engineers in Almeida and Kogut (1999) for instance. It used different methods to measure them: Ponds et al. (2010) proceeded by an estimation of a “knowledge production function”, following the method first used by Jaffe (1989), while Jaffe et al. (1993), and later on Thompson and Fox-Kean (2005) (in a refined way), used a matching method to assess localization of patent citations.

These considerations led to a reinforcement of the importance of MAR externalities, and a general support for cluster policies, as advocated by Porter (1998). Saxenian (1996) showed in a thorough analysis of the respective cultures of engineers in Route 128 in Boston, and Silicon Valley in California (two clusters in high technologies), that the first one plunged while the second was thriving because the culture of exchange between firms was much higher in the Silicon Valley. To our mind, this appeals for a double analysis of spillovers through local proximity on the one hand, and information flows in a network on the second hand.

Lately, many cluster policies have been led by public actors to foster innovation, and take advantage of agglomeration forces in less innovative sectors. For example, France first drew the “Local productive systems” policy from 1998, which was evaluated in Martin et al. (2011). This article showed that this policy had produced little results, and from 2005, the approach changed when the “pôles de compétitivité” policy was introduced. Fontagné et al. (2010) found it to be a policy selecting the most productive firms, in opposition with the latter policy that actually selected less productive firms. In these policies, the governments
subsidize common projects led by the firms entering the cluster. In our model, we will not replicate this setting, but rather consider the cluster as an area where any firm can settle if it chooses to incur the cost.

1.2 R&D collaboration networks

R&D cooperation networks have become a widespread phenomenon since the 1960s, as shown empirically in Hagedoorn (2002). These cooperative links consist of collaborations between companies that are independent, and enter into a formal agreement to share technological knowledge, build an R&D joint-venture or clinch a R&D collaboration contract without the creation of a dedicated common entity. These can appear between firms that are in totally separate markets, but also between firms that belong to the same industry (in a broad way). These cooperations have enjoyed a considerable growth since the 1960s, and mostly concern highly innovative sectors with high growth, such as biotechnologies (Powell et al. (1996), Roijakkers and Hagedoorn (2006)), the pharmaceutical industry (Staropoli (1998)), information technology (Hagedoorn and Schakenraad (1993)), chemicals (Ahuja (2000)), or nanotechnologies (Autant-Bernard et al. (2007)).

From a theoretical point of view, as we have seen, the seminal article by d’Aspremont and Jacquemin (1988) considered an oligopoly industry with spillovers, and named collaborations as the main phenomenon for spillovers, but did not modelize explicitly a network. The first theoretical analysis in a social network approach of these cooperations was done by Goyal and Moraga-Gonzalez (2001), who develop a model where firms have the possibility to form pairwise links in a first stage, and compete in an oligopoly market in a second stage. This article was then followed by many papers, among which Goyal et al. (2003), or Westbrock (2010) on welfare issues. König et al. (2007) and König et al. (2012) provide a different approach of the question since they use a model of “recombinant knowledge” in continuous time (innovation appears through the recombination of existing knowledge, and is then more likely when the amount of knowledge at the disposal of the firm is larger). A major contribution is also done in König et al. (2014), a working paper that proposes a complete analysis of the issue in terms of equilibria, public policy and empirical estimation.

1.3 Theoretical literature on social networks

The model we introduce is indeed very similar to the one developed by König et al. (2014), which is a generalized version of the issues raised by Goyal and Moraga-Gonzalez (2001). We build our analysis on a network that is formed ex-ante, which places our model in the class of games on networks, as reviewed by Jackson and Zenou (2012). We rely on the analysis led by Ballester et al. (2006), which is the major article in this field giving general results on equilibrium outcomes from the centrality of the players in games with strategic complementarities and linear quadratic payoffs. In this article, payoffs depend positively on own efforts but negatively on the square of efforts, and also depend on the structure of the network. Notably, the paper proves that in such a setting, if network effects are reasonably
high, there exists an interior solution for efforts which is the unique Nash equilibrium, that can be expressed as a function of the centrality.

A very important article on which we will rely is the one by Helsley and Zenou (2013). Indeed, this article is the first one that develops a two stage game where firms chose their location in the first place (either with a binary choice \( \{0, 1\} \) or in a continuum), and chose their level of interactions in a second stage. This article develops a method that allows to derive the equilibria using the centrality in the network, which we will follow in our game.

We will also assess the stability of the network in a final section, based on the concept of pairwise stability in networks developed by Jackson and Wolinsky (1996).

2 The model

2.1 Goals

We want to associate a local component to an R&D collaboration setting such as the one from König et al. (2014). Then, we also base our analysis on Helsley and Zenou (2013), which introduces a spatial component in a network analysis. In our framework, this spatial differentiation lies on the fact that spillovers tend to be localized, as evoked in the literature review.

Hence, we introduce a form of R&D cluster spillover that allows firms located in the cluster to get an \( \text{ex post} \) bonus level \( \psi_2 \) times the aggregate R&D efforts in the market of spillovers. The intuition behind this feature of the model is that firms enter clusters without knowing precisely what can be expected from it, and expect knowledge spillovers to happen later. Hence, they do not have an instantaneous control over these spillovers as it is the case with R&D cooperations that imply common R&D joint-ventures or common research projects, that could decrease the firm’s cost in the sense that collaborators efforts are captured directly on a current product. On the contrary, here, firms expect to follow what their competitors are currently doing, thus they improve their position for further discoveries, and get a bonus in profit that is actually interpreted here as an expected profit from future innovation. An analogy can be drawn with recombinant knowledge models (as Weitzman (1998) or König et al. (2011)): by entering the cluster, firms have a better view on the stock of knowledge accessible in this industry, and then increase their ability to recombine knowledge to produce a new innovation. This spillover adds up to a spillover firms get from bilateral cooperations (either intra periphery or cluster to periphery), that allow firms to capture the effort of their collaborators at a rate \( \psi_1 \).

We do not use a competition matrix defining several markets, and only consider that all firms compete in the same market, where they produce goods that are imperfectly substitutable. This is not an unrealistic assumption since clusters tend to be specialized poles of research, and then gather firms that produce goods in the same market, but that are
only partially substitutable, and so the firms are still likely to cooperate if the benefits of cooperation outweigh the costs in terms of increased competition.

2.2 Tools for network analysis

**Standard definitions** The network (or graph) $G$ of collaborations is defined as a set of nodes $\mathcal{N} = 1, \ldots, n$ and a set of links between these nodes $\mathcal{E}$. The network is here said to be undirected, meaning that each link is bilateral. Thus, there are $\frac{n(n-1)}{2}$ possible links. The degree $d_i$ is the number of links of a firm $i$. A firm is a neighbor of firm $i$ if it has a link with $i$. A walk of length $k$ from a firm $i$ to a firm $j$ is a way between these two nodes going through a path of $k$ links. A graph is connected if there is a path between any couple of nodes. A graph is said to be complete if all possible links exist, and empty if no link exists. The structure of the network is summed up into a $n \times n$ adjacency matrix. The links (or edges) are recorded in $A$, the adjacency matrix over $F$. Its elements are defined as:

$$
\begin{cases}
    a_{ij} = 1 & \text{if i and j are linked} \\
    a_{ij} = 0 & \text{otherwise}
\end{cases}
$$

It is worth noticing that firms are not considered to be linked with themselves. In any case, $a_{ii} = 0$, so that the diagonal of the adjacency matrix is composed of zeros. For example, even if the network is complete, the adjacency matrix will write: $A = uu' - I_n$, with $u$ the vector of ones and $I_n$ the identity matrix.

We also note that the $k^{th}$ power of the adjacency matrix $A^k$ records the walks of length $k$ in the network.

**The Bonacich centrality measure** In our developments, we will use extensively the notion of Bonacich centrality, introduced by Bonacich (1987). The idea of the Bonacich centrality is to provide a measure that counts the paths between player $i$ and all other players but that discounts the length of the path with a parameter $\phi \in [0, 1]$. This discounting allows to give less weight to walks that are very long, in which players are indirectly linked but can in fact be very distant. In our framework, it is obvious that direct technological links are much more valuable than indirect link in a path of length 4 for instance (path to a firm that is the neighbor of the neighbor of a neighbor). Thus, since $A^k$ records the walks of length $k$ in the network, we define the matrix:

$$
M(G, \phi) = \sum_{k=0}^{\infty} \phi^k A^k
$$

If and only if $\phi < \frac{1}{\lambda_{PF}}$ (i.e. when $(I - \phi A)$ is invertible), with $\lambda_{PF}$ the highest eigenvalue of $A$, this infinite geometric sum can be rewritten as:

$$
M(G, \phi) = (I - \phi A)^{-1}
$$
The (unweighted) vector of Bonacich centralities is then written as:

$$b_u(G, \phi) = M(G, \phi) \cdot u$$

where $u$ is the vector of ones.

We will also use a measure of centrality weighted by a vector (for example the vector of location that only gives a non-zero value to the walks between two firms in the cluster). It writes, with $l$ a weighting vector:

$$b_l(G, \phi) = M(G, \phi) \cdot l$$

The Bonacich centrality of individual $i$ can then be expressed as the sum of self-loops ($m_{ii}$s) and non self-loops ($\sum_{j \neq i} m_{ij}$).

### 2.3 The environment

We define the set of firms in the cluster $C$, and the set of firms in the periphery $P$, as $C \subset F$ and $P \subset F$, where $F$ is the set of firms in the network and $C \cap P = \{\emptyset\}$.

As usual in the social networks literature, we define $A$, the adjacency matrix over $F$ which records the links in our network, as:

$$a_{ij} = 1 \quad \text{if } i \text{ and } j \text{ have a collaborative link}$$
$$a_{ij} = 0 \quad \text{otherwise}$$

We define a two stage game:

- in the first stage, firms choose whether to pay the cost $\kappa$ to move into the cluster or not;
- in the second stage, as in König et al. (2014), firms choose their R&D efforts and their quantities.

Then, we assume that the structure of the network is predefined, and we study the impact of the introduction of the cluster on the stability of the pre-existing network and the incentives to sever pre-existing links (when firms do not belong both to the cluster for example) or incentives to create new ones (when firms both belong to the cluster and are not linked).

Following Singh and Vives (1984) and König et al. (2014), we introduce a representative consumer who obtains utility from consumption in the following way:

$$\bar{U}(\{q_i\}) = \alpha \sum_{i \in N} q_i^\alpha - \frac{1}{2} \sum_{i, j \in N} q_i^2 - \theta \sum_{i \in N} \sum_{j \neq i} q_i q_j$$

where $\theta$ denotes the substitutability of the goods, and $\alpha$ is the market size.
The consumer maximizes net utility:

\[ U = \bar{U} - \sum_i p_i q_i \]

This functional form gives rise to a linear demand structure. The inverse demand function is:

\[ p_i = \alpha - q_i - \theta \sum_{j \neq i} q_j \]

Following previous works by d’Aspremont and Jacquemin (1988) and König et al. (2014), the cost function of firm i is given by:

\[ c_i = \bar{c}_i - e_i - \psi_1 \sum_{j=1}^n a_{ij} e_j \]

This means that R&D collaborations are used to provide direct cost reduction, though the efficiency of partner firms’ efforts in reducing costs is of course inferior by a factor \( \psi_1 \) to own efforts. \( \psi_1 \) is the parameter that reveals the intensity of spillovers in this market.

The profit function is:

\[ \pi_i = (p_i - c_i)q_i + \psi_2 l_j \sum_{j=1}^n e_j - \gamma e_i^2 \]

with \( l_j = 1 \) if the firm locates in the cluster and 0 otherwise. The cost of effort is an increasing and concave function given by \( \gamma e_i^2 \).

As explained earlier, the cluster spillover is composed of \( \psi_2 \) multiplied by the sum of efforts in the market. Therefore, \( \psi_2 \) represents the factor of integration of technological knowledge, and may then capture elements of public policy such as common facilities in the cluster, labs, and universities spreading locally all the knowledge in this field. Thus, the sum of the efforts of the firms in the market is to be understood as the intensivity of research in this market and a stock of available knowledge. It will of course be of major importance in the amount of technological spillovers received, and very different from one industry to another. Thus, moving into the cluster, firms are enabled to use the total amount of knowledge produced in the market and increase their potential of innovation, i.e. their expected profits. This idea of innovation appearing from existing knowledge stems directly from recombinant knowledge models (Weitzman, 1998). We here chose to use the sum of all efforts for the cluster spillover, even though any function \( f(e_1, \ldots, e_n) \) could fit. From the interpretation developed of the cluster spillover, we chose to introduce a direct influence of firm’s own effort on this spillover. Indeed, since this term represents the industry intensivity of R&D, a single firm may have a huge influence in a market with a reduced number of players, and would then benefit a lot less from moving to the cluster, even from externalities linked to the presence of universities or amenities that are included in the \( \psi_2 \).
3 Equilibrium with exogenous location

Replacing $c_i$ by its expression, we can write:

$$\pi_i = (\alpha - \bar{c}_i)q_i - q_i^2 - \theta \sum_{j \neq i} q_i q_j + q_i e_i + q_i \psi_1 \sum_j a_{ij} e_j + \psi_2 l_i \sum_{j=1}^n e_j - \gamma e_i^2$$ (2)

We acknowledge that this specification for profits is not the most direct and intuitive one could think of. Indeed, we first tried to introduce a second matrix built on the same model as an adjacency matrix, where a 1 indicated that both firms located in the cluster, and a zero indicated that one of the two firms was in the periphery. Knowledge then spilled over in the same way as with explicit collaborations by cost reduction, but using the terms of a different matrix. Nevertheless, it was not possible to solve this model because we need to invert a sum of matrices and that we only have convenient properties on sums of two matrices in which one of them is rank 1. A second attempt was then to use a rank 1 matrix composed of row of ones for cluster firms and rows of zeros for peripheral firms, which provided a cost reduction proportional to the sum of aggregate efforts. This model did not allow us to conclude on equilibria using the methodology we will use here, because it led to fourth degree equations, whereas we are here able to conclude on equilibria thanks to the resolution of second degree equations that yield only one positive root. Therefore, it is clear that the choices made here for the cluster spillover is not the first-best as regards modelization design, though it should be thought of as a necessary compromise in a model using at the same time oligopoly, social networks, and location choice environments.

The first order condition on effort is:

$$\frac{\partial \pi_i}{\partial e_i} = q_i - 2\gamma e_i + \psi_2 l_i = 0$$ (3)

This yields the optimal effort:

$$e_i = \frac{1}{2\gamma}(q_i + \psi_2 l_i)$$ (4)

Therefore, for peripheral firm, $l_i = 0$, and the optimal effort is $e_i = q_i / 2\gamma$, as in König et al. (2014). When a firm belongs to the cluster, $l_i = 1$, and the second term of (4) is the ratio of the marginal benefit of efforts allowed by the cluster over the marginal cost of effort.

And the first order condition on quantities is:

$$\frac{\partial \pi_i}{\partial q_i} = \alpha - \bar{c}_i - 2q_i - \theta \sum_{j \neq i} q_j + e_i + \psi_1 \sum_{j=1}^n a_{ij} e_j = 0$$ (5)

Inserting optimal effort yields:

$$q_i = \frac{2\gamma(\alpha - \bar{c}_i)}{4\gamma - 1} - \frac{2\gamma\theta}{4\gamma - 1} \sum_{j \neq i} q_j + \frac{\psi_1}{4\gamma - 1} \sum_{j=1}^n a_{ij} q_j + \frac{\psi_2}{4\gamma - 1} l_i + \frac{\psi_1 \psi_2}{4\gamma - 1} \sum_{j=1}^n a_{ij} l_j$$ (6)
To alleviate the notations, we denote:

\[
\mu_i = \frac{2\gamma(\alpha - \bar{c}_i)}{4\gamma - 1}, \\
\rho = \frac{2\theta\gamma}{4\gamma - 1}, \\
\lambda_1 = \frac{\psi_1}{4\gamma - 1}, \\
\lambda_2 = \frac{\psi_2}{4\gamma - 1}, \\
\bar{q} = \sum_{i=1}^{n} q_i, \text{ the aggregate quantity produced in this market}
\]

Which gives the following demand function for firm \( i \):

\[
q_i = \mu_i - \rho[\bar{q} - q_i] + \lambda_1 \sum_{j=1}^{n} a_{ij}q_j + \lambda_2 (l_i + \psi_1 \sum_{j=1}^{n} a_{ij}l_j)
\]

Equation (7) can be decomposed into four terms which allow to underline all the features of our model: \( \mu_i \) is an idiosyncratic parameter that increases with the size of the market and decreases with own fixed costs, the second term is the oligopoly competition effect (hence decreasing in aggregate quantities), the third term is the positive effect gained through direct links in the network, and the fourth term is the positive effect on quantities of a firm locating in the cluster (which has a component based on direct spillovers, and a component that reveals the circulation of additional spillover through the network due to additional R&D effort of cluster firms). The following results follow the resolution procedure developed by König et al. (2014).

**Proposition 1.** We denote \( \phi_1 = \frac{\lambda_1}{1-\rho}, \phi_2 = \frac{\lambda_2}{1-\rho}, \text{ and } \mathbf{l}' = (I_n + \psi_1 \mathbf{A})\mathbf{l} \).

Then, using the unweighted, \( \mu \)-weighted and \( \mathbf{l}' \)-weighted Bonacich centralities, we can express the vector of quantities as a linear function of these centralities:

\[
\mathbf{q} = \frac{1}{1-\rho} \left[ b_{\mu}(G, \phi_1) - \rho \bar{q} b_{\mu}(G, \phi_1) + \lambda_2 b_{\mu}(G, \phi_1) \right]
\]

Written individually, each firm produces the following quantity:

\[
q_i = \frac{1}{1-\rho} \left[ b_{\mu,i}(G, \phi_1) - \rho \bar{q} b_{\mu,i}(G, \phi_1) + \lambda_2 b_{\mu,i}(G, \phi_1) \right]
\]
Proof. Rewrite (7) in matrix form. This gives the production vector:

\[
q = \frac{1}{1-\rho}\mu - \frac{\rho \bar{q}}{1-\rho}u + \frac{\lambda_1}{1-\rho}Aq + \frac{\lambda_2}{1-\rho}(l + \psi_1A)l
\]

\[
\left( I_n - \frac{\lambda_1}{1-\rho}A \right) q = \frac{1}{1-\rho}\mu - \frac{\rho \bar{q}}{1-\rho}u + \frac{\lambda_2}{1-\rho}(I_n + \psi_1A)l
\]

The matrix \((I_n - \phi_1A)\) is invertible whenever \(\phi_1 \lambda_{PF}(A) < 1\), where \(\lambda_{PF}(A)\) is the Perron-Frobenius eigenvalue, i.e. the highest eigenvalue of \(A\). In such a case:

\[
q = \frac{1}{1-\rho}(I_n - \frac{\lambda_1}{1-\rho}A)^{-1}\mu - \frac{\rho \bar{q}}{1-\rho}(I_n - \frac{\lambda_1}{1-\rho}A)^{-1}u + \frac{\lambda_2}{1-\rho}(I_n - \frac{\lambda_1}{1-\rho}A)^{-1}(I_n + \psi_1A)l
\]

Writing the weighted Bonacich centralities, this is equal to:

\[
q = \frac{1}{1-\rho}\left[ b_\mu(G, \phi_1) - \rho \bar{q} b_u(G, \phi_1) + \lambda_2 b_{\ell'}(G, \phi_1) \right]
\]

Proposition 2. Total quantities in the market can also be expressed as a function of weighted and unweighted centralities in the network:

\[
\bar{q} = \frac{\sum_{i=1}^{n}[b_{\mu,i}(G, \phi_1) + \lambda_2 b_{\ell',i}(G, \phi_1) + \lambda_1 b_{\ell,i}(G, \phi_1)]}{1 - \rho + \rho \sum_{i=1}^{n} b_{u,i}(G, \phi_1)}
\]  \hspace{1cm} (10)

Proof. Multiplying (8) by \(u'\) from the left yields:

\[
(1-\rho)u'q = (1-\rho)\bar{q} = \sum_{i=1}^{n}[b_{\mu,i}(G, \phi_1) - \rho \bar{q} b_{u,i}(G, \phi_1) + \lambda_2 b_{\ell',i}(G, \phi_1)]
\]

i.e.

\[
\bar{q} = \frac{\sum_{i=1}^{n}[b_{\mu,i}(G, \phi_1) + \lambda_2 b_{\ell',i}(G, \phi_1) + \lambda_1 b_{\ell,i}(G, \phi_1)]}{1 - \rho + \rho \sum_{i=1}^{n} b_{u,i}(G, \phi_1)}
\]

The results of these two propositions are very important because they provide a unique Nash equilibrium for the exogenous locations case, as showed in Ballester et al. (2006) for linear quadratic payoff structures as our. Therefore, we have been able to determine individual and aggregate quantities only using the structure of the network and the spatial repartition of firms. In the next section, we relax the assumption of exogenous location. This will allow, in a following section, to provide comparative statics of additional entries on quantities.
4 Introducing location choice

4.1 Incentives to move into the cluster

We allow firms to choose their location (0 for the periphery or 1 for the cluster) in a first stage of the game. We then solve the model by backward induction using the results from the second stage (with exogenous location) given previously. We found the formulation for $q$ as well as the equilibrium effort: $e_i = \frac{1}{2}(q_i + \psi_2 l_i)$.

We then introduce a parameter $\kappa$ that corresponds to the cost of moving into the cluster for a firm. To assess whether it moves or not into the cluster, a firm will then proceed to a cost-benefit analysis, and compare its profits with and without an entry. We consider the marginal choice of a firm $i$ when a number of firms have already entered (we allow ourselves to use both $\phi_1, \lambda_2, \psi_1, \psi_2$ couples of parameters to alleviate the notations).

We are here led to consider that firms do not consider the variation induced by their entry on $\bar{q}$, so that we have $q_i$ as a linear equation of weighted Bonacich centralities, with coefficients that do not involve the results of the 1st stage. This assumption is not as unrealistic as it seems in the first place. Indeed, it does not mean that $\bar{q}$ is constant ex post, but that each firm, in its cost-benefit analysis of entering in the cluster, takes into account the direct advantages of its entry, but do not take into account the increased competition induced by positive externalities on its competitors’ cost functions, and the consequent rise in $\bar{q}$ and decrease in prices. Thus, it means that the firm does not observe the cost function of its competitors. Moreover, this assumption does not mean either that the firm believes there is no competition effect: it may consider a cost of an increased competition when entering the cluster, but we integrate it into the $\kappa$ as a lump-sum cost. This is quite coherent with the assumption made previously that entering the cluster does not provide immediate cost reduction, but rather an increased future expected profit, for which it is very difficult to assess for the firm whether entering the cluster and benefiting from the spillover is not profitable compared to developing alone a new innovation (thus more costly but facing less competition when it is marketable). Marginally, this assumption may even be closer to the reality than assuming that the firm perfectly observes the effects of its entry on the future profit function of its competitors.

We denote $n_{ic}^i = \sum_{j=1}^{n} a_{ij} l_j^i$, i.e. the number of links the firm has with central firms (firms in the cluster). Recalling that we called $m_{ij}$ the terms of the matrix $M(\phi_1, G) = (I - \phi_1 A)$, equation (7) can be rewritten:

$$q_i = \frac{1}{1 - \rho} \sum_{j=1}^{n} m_{ij} [\mu_j + \lambda_2 l_j^i - \rho \bar{q}]$$

Following the resolution procedure developed in Helsley and Zenou (2013), we then decompose $q_i$s between self-loops (discounted walks in the network from $i$ to itself) and non
self-loops, so that we have:

\[
q_i = \frac{1}{1-\rho} \left( m_{ii}[\mu_i + \lambda_2(l_i + \psi_1 n_{lc}^i) - \rho \bar{q}] + \sum_{j \in C-i} m_{ij}[\mu_j + \lambda_2(1 + \psi_1 n_{lc}^j) - \rho \bar{q}] + \sum_{j \in P-i} m_{ij}[\mu_j + \lambda_2 \psi_1 n_{lc}^j - \rho \bar{q}] \right)
\]

Since we will focus on \( m_{ii} \),s, we denote \( r_i \) the non self-loops part, so that:

\[
q_i = \frac{1}{1-\rho} \left( m_{ii}[\mu_i + \lambda_2(l_i + \psi_1 n_{lc}^i) - \rho \bar{q}] + r_i \right)
\]

Proposition 3. The variation in own quantities for a firm moving to the cluster is:

\[
q_{i|C} - q_{i|P} = \phi_2 m_{ii}
\]

Therefore, the variation in quantities is summed up by this term \( \phi_2 m_{ii} \), \( \phi_2 \) being a parameter derived from \( \psi_2 \) (intensity of cluster spillovers) and \( m_{ii} \) the self-loops terms of the matrix \( M \), that is therefore a measure of the centrality of \( i \) in this network with a level \( \phi_1 \) of spillovers intensity.

For further use, we also write:

\[
q_{i|C}^2 = q_{i|P}^2 + \frac{2\lambda_2}{1-\rho} m_{ii} q_{i|P} + \frac{\lambda_2^2}{(1-\rho)^2} m_{ii}^2
\]

We use the result, proved in König et al. (2014), that when the optimal effort equates \( \frac{\psi_2}{2\gamma} \), firm i’s profits are given by \( \pi_i = (1 - \frac{1}{4\gamma}) q_i^2 \). Adding the effect of location of our model allows to write profits as a function of location and own quantities.

Proposition 4. Profits of firms conditional on their locations write:

\[
\pi_{i|P} = \left(1 - \frac{1}{4\gamma}\right) q_{i|P}^2 + \frac{\psi_1 \psi_2}{2\gamma} n_{lc}^i q_{i|P}
\]

\[
\pi_{i|C} = \left(1 - \frac{1}{4\gamma}\right) q_{i|C}^2 + \frac{\psi_1 \psi_2}{2\gamma} n_{lc}^i q_{i|C} + \frac{\psi_2}{2\gamma} [\bar{q} + \psi_2 n_c - \frac{\psi_2}{2}] - \kappa
\]

Precisions on the derivation of these profits are given in the appendices. We must then compare profits for firm i when it enters the cluster and when it does not. With \( \sum_{j=1}^n l_{ij}^a \) being the number of links firm i has with firms of the cluster, that we denoted \( n_{lc}^i \), and denoting \( n_c \) the total number of firms in the cluster, the firm will settle in the cluster if and only if:

\[
(14) - (13) \geq 0
\]
We recognize a second degree equation in $m_{ii}$. Thus,

\[ \Delta = \left( \frac{1}{(1-\rho)^2} r_i + \frac{\psi_1 \lambda_2 n_{lc}^i}{1-\rho} \right)^2 + 4 \frac{[\mu_i + \lambda_2 (\psi_1 n_{lc}^i + \frac{1}{2}) - \rho \bar{q}] \left[ \frac{2\gamma}{\psi_2} \kappa - \psi_1 n_{lc}^i r_i - \psi_2 n_c - \frac{\psi_2}{2} \right]}{(1-\rho)^2} \]

As soon as the second term of $\Delta$ is positive, it is clear that we have two real roots among which one is negative (and can then be dropped), and the second is positive and given by:

\[ m_{ii}^{(2)} = \frac{\sqrt{\Delta} - \left[ \frac{1}{(1-\rho)^2} r_i + \frac{\psi_1 \lambda_2 n_{lc}^i}{1-\rho} \right]}{2[\mu_i + \lambda_2 (\psi_1 n_{lc}^i + \frac{1}{2}) - \rho \bar{q}]} (1-\rho)^2 \]

For this term to be positive (and to have our expression be positive around the roots), we must set that:

\[ \mu_i + \lambda_2 (\psi_1 n_{lc}^i + \frac{1}{2}) - \rho \bar{q} \geq 0 \]

which is always verified since, when no firm goes into the cluster, the quantities produced are this term for each firm weighted with the elements of $M$ that are positive, and quantities must be non-negative whatever the number and the position of the links between the firms, so we have $\forall i, \mu_i + \lambda_2 \psi_1 n_{lc}^i - \rho \bar{q} \geq 0$. Moreover, we need to condition $\kappa$, such that $\sqrt{\Delta} > \frac{1}{(1-\rho)^2} r_i + \frac{\psi_1 \lambda_2 n_{lc}^i}{1-\rho}$:

\[ \kappa \geq \frac{\psi_2}{2\gamma} \left[ \psi_1 n_{lc}^i r_i + \psi_2 n_c - \frac{\psi_2}{2} \right] \]

If the set of parameters of the model satisfies these assumptions, we then have a threshold $m_{ii}^{(2)}$ in the self-loops of firm $i$. For convenience of the calculation, we would also set $4\gamma - 1 > 0$.

**Proposition 5.** When the cost of moving to the cluster is high enough, firm $i$ will settle into the cluster iff:

\[ m_{ii} > m_{ii}^{(2)} \]
With
\[ m_{ii}^{(2)} = -\left[ \frac{1}{(1-\rho)^2} r_i + \frac{\psi_1 \lambda_2 n_{lc}}{1-\rho} \right] + \sqrt{\Delta} \]
\[ \frac{2(\mu_i + \lambda_2(\psi_1 n_{lc}^i + \frac{1}{2})) - \rho q}{(1-\rho)^2} \]

Then, the higher the centrality of firm \( i \), the greater its incentive to settle into the cluster.

Note that the condition on \( \kappa \) is not mandatory: it just means that under this value, there is no positive threshold, so all the firms have an incentive to locate in the cluster, whatever their centrality.

### 4.2 Equilibria

We follow Helsley and Zenou (2013) on the definition of equilibria: we consider a network with a number of types equal to the number of players. Under the previous conditions on parameters, two agents with the same Bonacich centrality cannot make different location choices. Agents with a high Bonacich centrality cannot live in the periphery if other agents with a lower Bonacich centrality live in the center. Agents are ordered in function of their centrality, where agent \( 1 \) has the highest centrality and agent \( n \) the lowest.

The equilibria \( e \) when each firm has a different level of centrality (and then that there are \( n \) different types of players) are defined as follows:

- \( e = C \) refers to an equilibrium where all firms locate in the cluster;
- \( e = C_{n-1}P \) refers to an equilibrium where all firms but one (the one with the lowest Bonacich centrality) locate in the cluster;
- \( \ldots \)
- \( e = P \) refers to an equilibrium where all firms locate outside of the cluster.

As in Helsley and Zenou (2013), we denote the incentive to move into the cluster \( \Phi^e \). This function is then defined as:

\[
\Phi^e(m_{ii}) = \frac{\psi_2}{2\gamma} \left\{ \frac{1}{(1-\rho)^2} [\mu_i + \lambda_2(\psi_1 n_{lc}^i + \frac{1}{2})] - \rho q \right\} m_{ii}^2 + \left[ \frac{1}{(1-\rho)^2} r_i + \frac{\psi_1 \lambda_2 n_{lc}^i}{1-\rho} \right] m_{ii} + \left[ \frac{\psi_1 n_{lc}^i}{1-\rho} r_i + \frac{\psi_2 n_{c}}{2} \right] \right\}
\]

**Proposition 6.** When the number of types is equal to the number of firms, there is a unique equilibrium for a given value of \( \kappa \).

Since all firms have different types, we can draw a complete mapping over \( \kappa \) that allows to relate directly values of \( \kappa \) to values of the incentive for a firm to enter in the equilibrium refered as central where all firms are in the cluster. This complete mapping is provided in the appendices. We stick here to the conditions for the two polar cases (no firm in the cluster, all firms in the cluster).
Corollary 7. For the central equilibrium to hold we need the incentive for the least central player, \( n \), to be higher than the cost, i.e.

\[
\Phi^C(m_{nn}) > \kappa \tag{20}
\]

Corollary 8. The condition for a peripheral equilibrium to hold is:

\[
\Phi^C(m_{11}) - \frac{\lambda_2 \psi_2}{2 \gamma (1 - \rho)} \sum_{j=2}^{n} \left[ \psi_1 a_{j,j-1} \left( \frac{m_{j-1,j-1}^2}{1 - \rho} + m_{j-1,j-1} + m_{j-1,j} \right) + \frac{m_{j-1,j}^2}{1 - \rho} \right] < \kappa \tag{21}
\]

Assessing equilibria when some firms have the same type (same number of links) can seem tricky: indeed, as we saw, the number of firms in the cluster when the firms contemplate to enter is part of the incentive function \( \Phi \). However, this incentive function varies positively with the number of firms in the cluster, so if two firms have identical links and that one has an incentive to enter the cluster, then the other will have an even greater incentive to enter (since \textit{ex post}, \( n_c \) will have increased by 1). Therefore, deriving equilibria with a number of types inferior to the number of agents only requires to follow the same procedure with each step denoting a type and not a player, as stated in Helsley and Zenou (2013).

To our mind, a debate which remains open is whether firm’s own effort should count in the spillover it receives. Indeed, one could think of it as an increased access to other firms research, in which case the equilibria we have just derived would change a lot because the most central firms would not have a greater incentive to move in the cluster, since the potential amount of knowledge received thanks to the cluster would then be a decreasing function of own effort/centrality. In the interpretation made here of our cluster spillover term as an intensivity of the sector, we nevertheless believe that it should be included. For further research, a middle way could be to use a function of all efforts for the cluster spillover which gives a lower weight to own effort than to other firms’ efforts.

5 Comparative statics and welfare analysis

In this economy, aggregate welfare equates:

\[
W = U + \Pi \tag{22}
\]

where \( \Pi \) are aggregate profits and \( U \) is total utility.

Ideally, one would like to rank the various equilibria proposed in the previous section in terms of welfare, as a function of the structure of the network. Unfortunately, due to the relative complexity of our model, we did not manage to get tractable results on this part in the general case. However, it still seems important to propose comparative statics showing different effects that will influence the result on welfare of an additional entry in the cluster.
The process in the following subsections is the following: in an equilibrium with \( i \) firms in the cluster, what would be the effect on the whole economy if \( i+1 \) entered the cluster?

### 5.1 Effect of an entry on aggregate quantities

We want to know how quantities would evolve at the aggregate level when the firm \( i+1 \) decides to enter in the cluster. As we showed before:

\[
\bar{q} = \frac{\sum_{k=1}^{n} \left( \sum_{j=1}^{n} [\mu_j + \lambda_2 l_{j}^{i} m_{kj}] \right)}{1 - \rho + \rho \sum_{i=1}^{n} b_{u,i}(G, \phi_1)}
\]

Hence, the difference induced on \( l_{j} \) implies that the difference in total quantities is:

\[
\Delta \bar{q}^{G,P \rightarrow C_{i+1}P} = \frac{\lambda_2 \sum_{k=1}^{n} m_{k,i+1}}{1 - \rho + \rho \sum_{i=1}^{n} b_{u,i}(G, \phi_1)}
\]

\[
= \frac{\lambda_2 b_{u,i+1}(G, \phi_1)}{1 - \rho + \rho \sum_{i=1}^{n} b_{u,i}(G, \phi_1)}
\]

This equation shows that \( i+1 \)'s additional entry in the cluster always increases aggregate quantities, and that the magnitude of this effect is directly linked to the unweighted Bonacich centrality of this new entrant firm, and is linear in the rate of cluster spillovers \( \psi_2 \) (since \( \lambda_2 = \frac{\psi_2}{\psi_2 + 1} \)).

### 5.2 Effect of an entry on aggregate profits

Aggregate profits in the market are given by:

\[
\Pi = \sum_{i \in N} \pi_i
\]

In order to assess fully the competition effect triggered by an additional entry in the cluster, we fully consider changes in \( \bar{q} \) here.

The effect of an additional entry in the cluster on aggregate profits can be disentangled into five effects:

- an effect \( \Delta_1 \) on all firms due to an increase in \( \bar{q} \) (thus a decrease in prices);
- an effect \( \Delta_2 \) on all firms due to the increase in their \( l' \)-weighted bonacich centrality;
- an effect \( \Delta_3 \) that impacts the firms linked with the entrant (since the number of links with firms in the cluster increases by 1);
- an effect \( \Delta_4 \) that impacts firms in the cluster;
- an effect \( \Delta_5 \) that only impacts the entrant (this effect only records the difference with the 4 previous effects).
The first effect is that an entry impacts aggregate quantities, and the vector of quantities in (8) is negatively linked to aggregate quantities $\bar{q}$. Hence, it is directly linked with the effect of an additional entry on quantities we have just derived in the previous subsection. Recalling (23):

$$\Delta q^{C_i, \bar{P} \to C_{i+1} P} = \frac{\lambda_2 b_{u,i+1}(G, \phi_1)}{1 - \rho + \rho \sum_{i=1}^n b_{u,i}(G, \phi_1)}$$

Thanks to this effect on aggregate quantities, we can derive our second effect of an entry in the cluster. It can seem paradoxical that an additional entry has a positive effect on aggregate quantities but a negative effect on individual quantities. In fact, it just means that this effect is faced by each firm, which tend to reduce their quantities when competition increases, but that this effect is offset by positive effects of the entry on the entrant and the firms that already locate in the cluster and benefit from an increased spillover. Note that $q_i$s and $\bar{q}$ here refer to \textit{ex ante} values, of which we consider the variations.

As we have previously written, quantities at the optimum write:

$$q_i = \frac{1}{1 - \rho} \sum_{j=1}^n m_{ij} [\mu_j - \rho \bar{q} + \lambda_2' j]$$

Thus, the variation in quantities induced by an additional entry from the second effect (that applies to all firms) is:

$$\Delta_1 q_j^{C_i, \bar{P} \to C_{i+1} P} = \frac{1}{1 - \rho} \frac{-\rho \lambda_2 \sum_{k=1}^n m_{jk} \sum_{j=1}^n m_{k,i+1}}{1 - \rho + \rho \sum_{i=1}^n b_{u,i}(G, \phi_1)}$$

$$= \frac{-\rho \lambda_2 b_{u,j}(G, \phi_1) b_{u,i+1}(G, \phi_1)}{1 - \rho + \rho \sum_{i=1}^n b_{u,i}(G, \phi_1)}$$

We can then compute this second effect on aggregate profits:

$$\sum_{j=1}^n \Delta_1 \pi_j^{C_i, \bar{P} \to C_{i+1} P} = (1 - \frac{1}{4\gamma}) \rho^2 \phi_2^2 b_{u,i+1}(G, \phi_1)^2 \frac{\sum_{i=1}^n b_{u,i}(G, \phi_1)^2}{(1 - \rho + \rho \sum_{i=1}^n b_{u,i}(G, \phi_1))^2}$$

$$- 2(1 - \frac{1}{4\gamma}) \rho \phi_2 \bar{q} b_{u,i+1}(G, \phi_1) \frac{\sum_{i=1}^n b_{u,i}}{1 - \rho + \rho \sum_{i=1}^n b_{u,i}(G, \phi_1)}$$

$$+ \frac{\psi_1 \psi_2}{4\gamma} \rho^2 \phi_2 b_{u,i+1} \sum_{i=1}^n \frac{n_{i,j}}{1 - \rho + \rho \sum_{i=1}^n b_{u,i}(G, \phi_1)}$$

(24)

Since the first part of this first effect comes purely from the diminution in $\bar{q}$, we know that the second term in (25) dominates the first one. The third term is the indirect change produced by cluster through network links with firms that belong to the cluster (hence captured at the rate $\psi_1 \psi_2$).
**A second effect** is the positive impact of an entry on all firms’ quantities (thus profits) through the circulation of the additional spillover on the network. The effect of an entry on quantities indeed is negative through $\bar{q}$, but also has a positive component through the variation in quantities:

$$\Delta_2 q_{j}^{C_i \rightarrow C_{i+1}} = \frac{1}{1-\rho} m_{i+1,j} \lambda_2 = \phi_2 m_{i+1,j}$$

This second effect on aggregate profits can then be written as:

$$\sum_{j=1}^{n} \Delta_2 q_{j}^{C_i \rightarrow C_{i+1}} = (1 - \frac{1}{4\gamma}) \left[ \phi_2 \sum_{j=1}^{n} m_{j,i+1}^2 + 2\phi_2 \sum_{j=1}^{n} q_{j} m_{i+1,j} \right] + \frac{\psi_1 \psi_2 \phi_2}{2\gamma} \sum_{j=1}^{n} n_{lc}^j m_{j,i+1} \quad (26)$$

The sign of this effect is clearly positive.

**The third effect** is the specific effect on firm $i+1$ of its own entry. It is equal to:

$$\Delta_3 q_{i+1}^{C_i \rightarrow C_{i+1}} = \frac{\psi_2}{2\gamma} \left[ \bar{q} + \psi_2 (n_{c} + 1) - \frac{\psi_2}{2} \right] - \kappa \quad (27)$$

Note that, if the effect on firm $i+1$ was positive, it would yield an equilibrium where firm $i+1$ is in the cluster, which is incompatible with the equilibrium where only $i$ firms belong to the cluster. That is to say, if the optimal size of the cluster was above the market size, the government could choose to subsidize entries in the cluster so that several firms experience a positive net effect of an entry whereas it would normally have been negative.

**The fourth effect** is the positive (additional) effect experienced by firms that already belonged to the cluster (since, as we saw, the structure of their profits is different) from an increase in $\bar{q}$ and in $n_c$. This effect writes:

$$\sum_{j=1}^{n} \Delta_4 q_{j}^{C_i \rightarrow C_{i+1}} = \frac{\psi_2}{2\gamma} \left[ (i+1) \frac{\lambda_2 b_{u,i+1}}{1-\rho + \rho \sum_{i=1}^{n} b_{u,i}} + \psi_2 \right] \quad (28)$$

This effect is clearly positive.

**The fifth effect** is due to the fact that quantities produced by the firm positively depend on the number of links they have with firms in the cluster, which potentially increased by 1.
The total of this effect then is:

\[ \sum_{j=1}^{n} \Delta_5 \pi_j^{C_iP \to C_{i+1}P} = \frac{\psi_1 \psi_2}{2\gamma} \sum_{i=1}^{n} a_{i+1,j} q_j^2 \]  

This effect is also positive.

5.3 Effect of an entry on aggregate utility

We defined earlier the gross utility of consumption \( \bar{U} \). Net utility is then obtained with:

\[ U = \bar{U} - \sum_i p_i q_i \]

Inserting the inverse demand into the utility function of the representative consumer yields:

\[ U = \frac{1}{2} \sum_{i \in N} q_i^2 + \frac{\theta}{2} \sum_{i \in N} \sum_{j \neq i} q_i q_j \]

To simplify the algebra, we make the assumption that goods are perfectly substitutable in this economy, hence that \( \theta = 1 \). This is not so much an assumption as a lower bound for the effect that an additional entry will have on welfare\(^3\). This assumption simplifies the algebra:

\[ U = \frac{1}{2} \left( \sum_{i=1}^{n} q_i \right)^2 = \frac{1}{2} q^2 \]  

(30)

As we evidenced in the previous subsection, an additional entry of firm \( i + 1 \) has an effect on quantities given by (23). Thus,

\[ \Delta_6 U = \frac{1}{2} \left[ \frac{\lambda_2 b_{u,i+1}(G, \phi_1)^2}{[1 - \rho + \rho \sum_{i=1}^{n} b_{u,i}(G, \phi_1)]^2} + \frac{\lambda_2 b_{u,i+1}(G, \phi_1)}{1 - \rho + \rho \sum_{i=1}^{n} b_{u,i}(G, \phi_1)} \right] \]

(31)

The effect for the representative consumer is clearly positive, because he benefits from the increase in aggregate quantities, that induces a decrease in prices and allows him to consume more. In the other polar case, where goods are not substitutable (\( \theta = 0 \)), \( U = \frac{1}{2} \sum_{i=1}^{n} q_i^2 \), which is inferior to the previous form of utility, and then allows for a reduced effect of an additional entry. Indeed, substitutability is inferior to one, therefore, for equal increases in quantities, the increase in competition is lower that when goods are perfectly substitutable, and prices decrease less, leading to a lower effect on aggregate utility (but to a higher effect on aggregate profits).

\(^3\)Indeed, with perfectly substitutable goods, the first effect (the strengthened competition effect) is at its highest, and thus the effect on an additional entry is the most likely to be negative. Thus, we rather make this assumption to study a limit case than to assess a microfounded and general feature of firms in clusters.
5.4 Effect on profits of non-entrants

One may wonder what the effect of an additional entry in the cluster is on non-entrants. Indeed, as one could state from the disentangled effects of an entry, firms that do not enter face as hard as others the increased competition effect, but do not benefit from additional positive effects that the most central firms get through their presence in the cluster. Thus, it is not difficult to imagine a situation in which the entry in the cluster is subsidized, so that all the central firms (which are already the “winners”) get positive effects from the cluster (and on top of that get subsidized for their entry), but less central firms only get a residual of the spillovers through their potential links with firms that fully participate to the cluster, whereas they face a harshly increased competition. Thus, the effect of having a $i + 1^{th}$ entry on non-entrant firms’ quantities writes:

$$\Delta_{C_i \rightarrow C_{i+1}} q_{j|i+1} = -\rho \phi_2 \frac{b_{u,j}(G, \phi_1) b_{u,i+1}(G, \phi_1)}{1 - \rho + \rho \sum_{i=1}^{n} b_{u,i}(G, \phi_1)} + \phi_2 m_{i+1,j}$$

(32)

This variation in quantity is negative as soon as

$$m_{i+1,j} < \rho b_{u,i+1}(G, \phi_1) \frac{b_{u,j}(G, \phi_1)}{1 - \rho + \rho \sum_{i=1}^{n} b_{u,i}(G, \phi_1)}$$

(33)

Therefore, injecting this variation in (13) yields:

$$\Delta_{C_i \rightarrow C_{i+1}} \pi_{j|i+1} = \left(1 - \frac{1}{\gamma}\right) \left((\Delta q_{j|i+1})^2 + 2\Delta q_{j|i+1} q_{j|i+1}^* + \frac{\psi_1 \psi_2}{2\gamma} (n_{c} + a_{i+1,j}) \Delta q_{j|i+1}\right)$$

(34)

This variation in profits is negative as soon as the variation in quantities, given in (33), is negative. The only way a non-entrant may benefit from the entry of firm $i + 1$ is to have a high $m_{i+1,j}$, that represents the paths between $i + 1$ and $j$ in the network. That is to say, the non-entrant firms may individually benefit from an additional entry if they have a particular linkage with the entrant. However, since the most central firms have a higher incentive to enter, the fact that the firm $j$ does not enter means that it is not very central in the network, and thus is unlikely to be intensely linked with $i + 1$.

This will then depend on the assortativity of the network. Assortativity is the likeliness that firms with a high degree (number of links) are mostly linked with other firms with a high degree. So, if the network is very assortative and that all high degree firms enter, they will only be linked with each other, and the remaining firms of low degree will also very probably be linked between them, but to a lower extent. Thus, $i + 1$’s entry will not benefit to non-entrant firms because of too few links with the central firms that would have allowed them to benefit from the spillovers of the central firms, but it will harm them due to the advantage gained by their competitor $i + 1$, and the $i$ other competitors that may

\[\text{except if } i + 1 \text{ has a low enough centrality so that it is linked only with non-central firms}\]
benefit from cluster spillovers. With positive assortativity, the closer \( j \) is from \( i + 1 \) (in terms of degree), the more likely is the effect to be positive, because \( j \) will then have a higher degree.

One may also notice that when the net effect of an additional entry is negative on the firms belonging to the cluster, it is even more negative on firms belonging to the periphery. A way to have a positive net effect of an additional entry on non-entrants can be to have low substitutability of goods, or to have a very low assortativity of the network, meaning that firms with a high number of links (entrants) are very likely to be linked with firms with a low number of links, and then to allow a “distribution” of the cluster spillover.

These results give an interesting intuition in terms of cluster public policies: if the \( \text{ex ante} \) network between the firms in the targetted market is very assortative, the policy is likely to be welfare-improving but to hurt the less central firms, and thus may very well hurt employment, for example, or another variable that can enter the utility function of a social planner. For instance, as shown empirically in France by Martin et al. (2011), the less productive (and surely in our model the less central firms) tend to be localized in regions that suffer a lot, and in which public actors absolutely want to defend employment for territorial equity reasons. This suggests that, empirically, the designers of such cluster policies may want to avoid absolutely harming effects on the less productive firms.

6 Stability of the network

We now want to assess whether the network, that we previously considered as exogenous, is stable. Since standard equilibrium concepts (such as Nash equilibrium) do not fit to analyse the stability of a network, we will use the (now) standard tool of pairwise stability in networks, as first introduced by Jackson and Wolinsky (1996). A network is said to be pairwise stable when no node (here no firm) has an incentive to sever existing links because its profits would be higher without these links, and when no couple of firms has a common incentive to create a link between them.

Formally, a network \( G \) is said to be **pairwise stable** if:

1. For all link \( ij \in G \), \( \pi_i^i(G) \geq \pi_i^i(G_{-ij}) \) and \( \pi_j^j(G_{-ij}) \geq \pi_j^j(G_{-ij}) \);

2. For all link \( ij \notin G \), if \( \pi_i^i(G_{+ij}) > \pi_i^i(G) \), then \( \pi_j^j(G_{+ij}) < \pi_j^j(G) \).

This definition means that any player is free to sever a link if she does not get a benefit from it, and that a link can only be created if two players agree on it. It is clear then that we consider single deviations from an exogenous network rather than a complete process of network formation, and that the outcome will be fundamentally different, since the players only need an ex-post incentive to keep links, rather than an ex-ante incentive to create one. We do not chose to introduce non-negligible costs of formation, as some recent articles did (König et al., 2011).
Moreover, we are only able to address stability when we do not allow firms to change their choice in the first stage of the game (location choice). Indeed, if firms were able to change their mind after some links are created or severed, it would very probably change the game equilibria and point out stable and unstable equilibria, which we do not address here. Empirically, our analysis would rather correspond to an explicit innovation cluster policy, like the one that took place in France from 2005, called “pôles de compétitivité”, for which a tender calling for participation was issued, after which firms wanting to participate must wait for a reorganization of the policy (as it happened in France in 2009).

6.1 Incentives to create new links

We first consider the incentives to create a link between firms that belong to the periphery. The fact of creating a link between $i$ and $j$ changes $i$’s and $j$’s quantities. It also changes aggregate quantities, that are used to compute the change in quantities induced by the creation of the link. As we expressed $\bar{q}$ as a function of Bonacich centralities, it is quite difficult to derive the change in $\bar{q}$ when Bonacich centralities change. Indeed, assuming that a link is formed between $i$ and $j$, the new adjacency matrix $A'$ can be written as:

$$A' = A + L$$

with

$$L = \begin{pmatrix}
0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \vdots & 0 & \ldots & 1 & \ldots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

Therefore, the new vector of Bonacich centralities writes:

$$b_u(G, \phi_1) = M'.u = (\sum_{k=0}^{\infty} \phi_1^k A'^k)u$$

And

$$(A')^k = \sum_{j=0}^{k} \binom{k}{j} A^j L^{k-j}$$

So, this is heavy to compute and we here choose to make a simplification to assess the stability of the network. Thus, we will consider that for firms other than $i$ and $j$ which may form or break a link, the effect of competition is exactly compensated by the positive effect.
of additional spillovers. This writes:

$$\sum_{k \neq \{i,j\}} \Delta^{G+ij} q_k = 0$$

As long as this does not “disconnect” a part of the network (i.e. as long as the link does not have an important strategic value), this assumption is not very strong and should rather be thought of as an approximation.

Thus, in this framework, we consider the variation of aggregate quantities through the modification of a link (creation or severance) as being only composed of the variations of quantities of the two nodes that are concerned by this link. Formally:

$$\Delta^{G+ij} \bar{q} = \Delta q_i + \Delta q_j$$

**Proposition 9.** When both firms belong to the periphery, \(l_i = l_j = 0\), and the condition to form a link sums up to:

$$\begin{cases} q_j \geq \rho q_i \\ q_i \geq \rho q_j \end{cases}$$

**Proof.** It appears clearly in (13) that profits of peripheral firms are increasing in own quantities. Thus, a necessary and sufficient condition for two peripheral firms to create a link between them is to have their quantities increase with this link creation.

The variation of the quantities of two firms (possibly belonging to the cluster) writes:

$$\begin{cases} \Delta q_i = \lambda_1 q_j + \lambda_2 l_j - \rho \Delta q_j \\ \Delta q_j = \lambda_1 q_i + \lambda_2 l_i - \rho \Delta q_i \end{cases}$$

$$\iff \begin{cases} \Delta q_i = \frac{1}{1-\rho^2} [\lambda_1 q_j + \lambda_2 l_j - \rho \lambda_1 q_i - \rho \lambda_2 l_i] \\ \Delta q_j = \frac{1}{1-\rho^2} [\lambda_1 q_i + \lambda_2 l_i - \rho \lambda_1 q_j - \rho \lambda_2 l_j] \end{cases}$$

It appears that the condition in 9 is going to be slack on one part. Indeed, ordering the two firms so that \(q_i > q_j\), the condition \(q_i \geq \rho q_j\) will always be satisfied, and the condition to satisfy turns out to be \(q_j \geq \rho q_i\). Thus, if the ratio \(\frac{q_j}{q_i}\) of quantities produced by the two peripheral firms is superior to the ratio \(\rho\), the network is not stable because these two firms have an incentive to form a link \(ij\). This holds under the assumption that the effect of this link is neutral for firms in \(N - \{i, j\}\). When this is not true and that the effect is in average beneficial to the firms (i.e. makes \(\bar{q}\) increase), the condition to have a link formation is less restrictive (and conversely).

When this condition is met for all pairwise ratios, all firms in the periphery have an incentive to be linked with each other. Moreover, firms in the cluster have the same profits.
plus positive bonuses from links, thus the condition on the ratio of the quantities produced by peripheral firms over quantities of the central firms is more restrictive than the actual condition to obtain a link. Therefore, 

\[ \forall i, j, \text{ with } q_i > q_j: \quad \frac{q_j}{q_i} > \rho \]  

is a sufficient condition to get a complete network (the complete network is the network where all \( \frac{n(n-1)}{2} \) possible links are formed).

Apart from the polar case of the complete network, we would also like to analyse the stability of links when both firms belong to the cluster. In this case, the condition is not as simple as the previous one. Indeed, firms in the cluster get a bonus which is an increasing function of \( \bar{q} \), so we need to consider the change in profits:

\[
\Delta^{G+i+j}_{\pi_{i|C}} = \Delta q_i \left( (1 - \frac{1}{4\gamma}) \Delta q_i + \frac{\psi_1 \psi_2}{2\gamma} (n_{i|c}^i + l_j) + \frac{\psi_2}{2\gamma} + \frac{\psi_2}{2\gamma} \Delta q_j \right) \]  

We know that if the variation in quantity is positive, the difference will also be positive and the link will be created because this is a condition under which all possible links will be created.

When both firms belong to the cluster, with \( q_i > q_j \), the condition is less restrictive than between two peripheral firms and writes:

\[ q_j \geq \rho q_i - (1 - \rho) \frac{\lambda_2}{\lambda_1} \]

When only one firm belongs to the cluster, the condition is more restrictive than between two peripheral firms and writes:

\[ q_j \geq \rho(q_i + \frac{\lambda_2}{\lambda_1}) \]

What we also want to assess is the possibility that a link is created when \( \Delta q_i < 0 \), i.e., the possibility that a link is created even though the competition effect is intense enough to lead to a decrease in the quantity produced of the most productive firms of the two (because the spillover is higher for the less productive/central firm, that will always have its quantities increase). In this case, rewriting (37):

\[
\Delta^{G+i+j}_{\pi_{i|C}} = \Delta q_i \left( (1 - \frac{1}{4\gamma}) \Delta q_i + \frac{\psi_1 \psi_2}{2\gamma} (n_{i|c}^i + l_j) + \frac{\psi_2}{2\gamma} + \frac{\psi_2}{2\gamma} \Delta q_j \right) \]  

In such a case, a necessary (but not sufficient) condition is:

\[
\frac{(1 - \frac{1}{4\gamma})}{(1 - \rho)^2} [\lambda_1 q_j + \lambda_2 l_j - \rho \lambda_1 q_i - \rho \lambda_2 l_i] + \frac{\psi_1 \psi_2}{2\gamma} (n_{i|c}^i + 1) + \frac{\psi_2}{2\gamma} > 0 \]

(otherwise the product of the two is positive).
Assuming that this condition is met in our setting, a sufficient condition to form a link is then to have a first term of (38) that is lower in absolute value (and which is negative) than the second (which is positive). This means that the bonus increasing in $\bar{q}$, which is obtained thanks to the cluster, offsets the competition effect induced by the new link. However, considering the complexity of the condition and the number of parameters involved, it is very difficult to assert that this condition is restrictive or that it is not, because this will depend both on the set of parameter values chosen (especially the intensivity of the cluster spillovers), and on the values of $q_i$s, i.e. on the structure of the exogenous ex ante network.

Another interesting feature to check is whether the stable network will be assortative or not. Profits of a central firm (with our assumption of neutrality) are:

$\Delta^{G+ij}_{\pi_i|C} = \frac{1 - \frac{1}{4\gamma}}{(1 - \rho)^2} [\lambda_1 q_j + \lambda_2 l_j - \rho \lambda_1 q_i - \rho \lambda_2]^2 + \frac{\psi_1 \psi_2}{2\gamma(1 - \rho)^2} [\lambda_1 q_j + \lambda_2 l_j - \rho \lambda_1 q_i - \rho \lambda_2]$

$$+ \frac{\psi_1}{2\gamma(1 - \rho)^2} [\lambda_1 q_j + \lambda_2 l_j - \rho \lambda_1 q_i - \rho \lambda_2]$$

The derivative is:

$$\frac{\partial \Delta^{G+ij}_{\pi_i|C}}{\partial q_j} = \frac{(1 - \frac{1}{4\gamma})}{(1 - \rho)^2} 2\lambda_1 [\lambda_1 q_j + \lambda_2 l_j - \rho \lambda_1 q_i - \rho \lambda_2] + \lambda_1 \frac{\psi_1 \psi_2}{2\gamma(1 - \rho)^2} (n_{ic}^j + l_j) - \frac{\psi_2 \lambda_1}{2\gamma(1 - \rho)^2}$$

Hence

$$\frac{\partial \Delta^{G+ij}_{\pi_i|C}}{\partial q_j} \geq 0$$

$$\Leftrightarrow q_j \geq \rho q_i + \frac{\psi_1 \psi_2}{\psi_1} [1 + (1 - \rho)^2] - l_j [\psi_2(1 - \rho)^2 + \frac{\psi_2}{\psi_1}] - \psi_2(1 - \rho)^2 n_{ic}^j$$

(39)

This means that there is a threshold in the partner’s production from which it is preferable to be linked with a bigger producer (under this threshold, the competition effect is too strong and the firm prefers smaller producers). However, it is quite likely that under this production threshold, firm $j$ would not satisfy the participation constraint for $i$, and the absence of link between the two firms would be stable because their productions are too different. However, above the threshold, the most central firms in the cluster would rather create links with other very central firms (if they are not linked yet). This will yield very assortative stable networks, because the range of cases will go from empty stable networks to networks where a subset of firms belonging to the cluster are linked, or where all the firms in the cluster are linked, and some are linked with peripheral firms. The only case when the network will not be hyper-assortative is when there are possibly stable links between central and peripheral firms under the threshold shown in (39). Apart from this case, the assortativity property should preserve the hierarchy of centralities/quantities that led to the equilibrium of our game, so it should not make its outcome unstable (firms do not get an incentive to move out of the cluster after having joined it because the network changed).

We then need to study the incentives structure when one firm belongs to the cluster and the
other does not.

A third question is to assess the possibility link formation between a firm of the cluster and firms of the periphery. The condition for the creation of such a link between a firm $i$ in the cluster and a firm $j$ in the periphery is:

\[
\begin{cases}
\Delta^{G+ij} \pi_{i|C} \geq 0 \\
\Delta^{G+ij} \pi_{j|P} \geq 0
\end{cases}
\]

As we saw, it is sufficient for $j$ to see its quantities increase. The greater $q_i$, the more likely this condition is to be verified. Therefore, it is clear that peripheral firms will always have an incentive to form links with firms that produce high quantities (and that are very central), apart when the combination of parameters and the ex ante network lead to the empty network as the only stable network. The binding constraint will then be the one which ensures that $i$ has an incentive to create a link with a peripheral firm. For $i$, the condition was exposed previously.

- In the case where \( \frac{\partial \Delta^{G+ij} \pi_{i|C}}{\partial q_i} \geq 0 \), firm $i$ will form links with peripheral firms only if it is already steadily linked to all the central firms;
- In the case where \( \frac{\partial \Delta^{G+ij} \pi_{i|C}}{\partial q_j} < 0 \), the stable network may be non-assortative, and central firms may want to avoid linkages with other central firms from which they face harsher competition, and prefer to link with firms that produce low quantities, even if they capture a lower spillover because their effort - which is proportional to quantities for peripheral firms - is lower.

### 6.2 Incentives to sever links

For peripheral firms, the condition to sever links is the inverse of the one for link creation. Indeed, since the firm producing the lowest quantities always wants to create a link, the condition for creation resumes to a single condition. Moreover, the severance of a link only requires that one firm wants to delete it, and the firm producing the most will always be the first one wanting to sever the link. Therefore, when $i$ and $j$ are two peripheral firms with a link, and $q_i > q_j$, firm $i$ will sever the link as soon as:

\[
\frac{q_j}{q_i} < \rho
\] (40)

For firms which belong to the cluster, however, the incentive to sever links changes in two opposite ways. With peripheral firms, the ratio $\frac{q_i}{q_j}$ is lower, but they would also lose the bonus increasing in $\bar{q}$, so there is a trade-off that allows for a stable situation with links between two peripheral firms and links between a central and a peripheral firm coexisting, and even with remaining links between the center and the periphery when all the intra-periphery links would have been severed (under restrictive parametric conditions, however). As for the links between two central firms, the condition is similar to the previous one,
since (40) is a sufficient but non necessary condition, and there can be links which decrease quantities but which increase profits or keep them unchanged.

7 Public policy

7.1 Increase in the intensity of cluster spillovers

As we have suggested before, \( \psi_2 \) may be considered as a public policy parameter. Indeed, it indicates the rate at which aggregate efforts diffuse on profits of firms when they are embedded in the cluster. Looking at how clusters are created empirically, it is obvious that public actors tend to create a positive environment for further innovation, an ecosystem that includes common facilities, labs, universities with specialized programs, etc. Therefore, it is natural to address what happens in our economy when \( \psi_2 \) varies, i.e. when public actors decide to create an indirect subsidy by investing in facilities or in education and research structures inside the cluster, to increase the efficiency of spillovers inside this area.

Proposition 10. The effect on aggregate quantities of an increase in \( \psi_2 \) is:

\[
\frac{\partial \bar{q}}{\partial \psi_2} = \frac{\sum_j b_{v,j}(G, \phi_1)}{(4\gamma - 1)(1 - \rho + \rho \sum_j b_{u,j}(G, \phi_1))}
\] (41)

Under the conditions on parameters we adopted previously \( (4\gamma - 1 > 0, 1 - \rho > 0) \), this will obviously be positive. However, the magnitude of this effect will be very small when very few firms are in the cluster because the weights in the \( l' \) weighted centrality (since vector \( l' \)'s components are small) are small compared to the sum of the unweighted Bonacich centralities.

Proposition 11. The effect on quantities produced by a firm \( i \) of an increase in \( \psi_2 \) is:

\[
\frac{\partial q_i}{\partial \psi_2} = \frac{1}{(1 - \rho)(4\gamma - 1)} \left( b_{v,i}(G, \phi_1) - \rho \frac{\sum_j b_{v,j}(G, \phi_1)}{(1 - \rho + \rho \sum_j b_{u,j}(G, \phi_1))} \right)
\] (42)

This shows, as in the previous results, that there will be a trade-off between the direct advantage gained through an increase in \( \psi_2 \) (even if the firm does not belong to the cluster, because the level of effort of its partners that belong to the cluster increases), and the competition effect triggered by the increase in total quantities. Therefore, the effect of an increase in \( \psi_2 \) on quantities will be positive iff:

\[
b_{v,i}(G, \phi_1) > \rho \frac{\sum_j b_{v,j}(G, \phi_1)}{(1 - \rho + \rho \sum_j b_{u,j}(G, \phi_1))}
\] (43)

The term in the right-hand side is an increasing function of \( \rho \), so, the higher substitutability between goods, the higher the threshold for the effect of an increase in \( \psi_2 \) on quantities to be positive.
Another feature of this threshold on \( b_{i}(G, \phi_{1}) \) that appears is that the right hand side of the inequality, when the network is dense enough, is close to a ratio of the sum of centralities weighted by \( l' \) over the sum of unweighted centralities. Considering Bonacich centralities weighted by the location vector means that paths between two firms of the cluster count a lot more than direct links between very peripheral firms. Therefore, the denser the network weighted by locations, the higher the threshold on \( i' \)'s own centrality weighted by \( l' \) in (43) to have a positive effect on its quantities. In other words, the denser the location weighted network, the more central in the location weighted network \( i \) will have to be to get a positive effect on its quantities from an increase in \( \psi_{2} \). If a firm is somehow linked with the cluster but that in average, other firms are more linked to the cluster than this firm, the overall effect of an increase in \( \psi_{2} \) will be a decrease in quantities due to a very strong competition effect. This result converges with the previous result given on the effect on an additional entry in the cluster on firms which do not belong to the cluster, that may also be negative. A cluster policy may then be welfare-improving, but hurt a lot the less productive and less central firms in the market because it strengthens the competition even when knowledge spills over to peripheral firms to a certain extent (through collaboration links).

**Proposition 12.** Denoting \( q'_{i} = \frac{\partial q_{i}}{\partial \psi_{2}} \):

1. The derivative of the profits of a peripheral firm with respect to \( \psi_{2} \) is:

\[
\frac{\partial \pi_{i|P}}{\partial \psi_{2}} = 2\left(1 - \frac{1}{4\gamma}\right)q'_{i}q_{i} + \frac{\psi_{1}n_{lc}^{i}}{2\gamma}(q_{i} + \psi_{2}q'_{i})
\]

(44)

2. For firms in the cluster, the effect of an increase in \( \psi_{2} \) is much more likely to be profitable. Indeed, the variation of their profits writes:

\[
\frac{\partial \pi_{i|C}}{\partial \psi_{2}} = 2\left(1 - \frac{1}{4\gamma}\right)q'_{i}q_{i} + \frac{\psi_{1}n_{lc}^{i}}{2\gamma}(q_{i} + \psi_{2}q'_{i})
\]

\[
+ \frac{\psi_{2}\sum_{j}b_{i,j}(G, \phi_{1})}{2\gamma(4\gamma - 1)(1 - \rho + \rho \sum_{j}b_{i,j}(G, \phi_{1}))} + \frac{\bar{q}}{2\gamma} + \frac{\psi_{2}n_{c}}{\gamma} - \frac{\psi_{2}}{2\gamma}
\]

(45)

Proposition 12 1. is equivalent to stating that an increase in \( \psi_{2} \) lowers the profits of peripheral firms as soon as:

\[
q'_{i} \leq -\frac{\psi_{1}n_{lc}^{i}}{(4\gamma - 1)q_{i} + \psi_{1}\psi_{2}n_{lc}^{i}}
\]

One should also notice that in proposition 12 2., we know that \( q'_{i|C} \) is also much more likely to be positive than \( q'_{i|P} \), so that even for the less central firm in the cluster (the "last entrant"), the effect of an increase in \( \psi_{2} \) on profits should be positive under almost all parametric settings.

An important variable to assess the effect on welfare of variations in \( \psi_{2} \) is the cost for the government of setting \( \psi_{2} \) at a certain level, which we denote \( \chi(\psi_{2}) \). The function \( \chi \) is
increasing and convex: costs increase with $\psi_2$, and these costs exhibit decreasing returns (hence $-\chi(\psi_2)$ is concave). There is a natural value of $\psi_2$ for which $\chi(\psi_2) = 0$, i.e. the government does not intervene. It corresponds empirically to what would be called “natural clusters”, that is to say a natural circulation of information at a local scale, without necessitating public infrastructures. We hereby consider that the government finances its intervention through taxation of the representative consumer, so that the weight of public intervention is completely endured by the representative consumer.$^5$

**Proposition 13.** In such a case:

1. A general expression of net utility is:

$$U = \frac{1 - \theta}{2} \sum_i q_i^2 + \frac{\theta}{2} \bar{q}^2 - \chi(\psi_2)$$

(46)

2. When goods are perfectly substitutable, net utility simplifies to:

$$U = \frac{1}{2} \bar{q}^2 - \chi(\psi_2)$$

(47)

3. The effect on aggregate utility of a variation in $\psi_2$ then is:

$$\frac{\partial U}{\partial \psi_2 |_{\theta=1}} = \frac{\sum_j b_{v,j}(G, \phi_1)}{(4\gamma - 1)(1 - \rho + \rho \sum_j b_{u,j}(G, \phi_1))} \bar{q} - \chi'(\psi_2)$$

(48)

This allows us to draw a possibility of public action: the government could choose to finance clusters such that the effect on consumers is neutral. This point is reached when the marginal cost of increasing $\psi_2$ equates the marginal benefit of $\psi_2$ for consumers:

$$\chi'(\psi_2) = \frac{\sum_j b_{v,j}(G, \phi_1)}{(4\gamma - 1)(1 - \rho + \rho \sum_j b_{u,j}(G, \phi_1))} \bar{q}$$

(49)

This will again depend positively on the relative density of the location weighted network with respect to the unweighted density of the network.

7.2 Linear R&D effort subsidy

A very standard tool for policy makers to foster innovation is to use a linear R&D effort subsidy, that may be a direct subsidy, or indirect tax incentives$^6$. We introduce a direct linear subsidy on R&D efforts, that changes the profits structure to:

$$\pi_i = (\alpha - c_i)q_i - q_i^2 - \theta \sum_{j \neq i} q_i q_j + q_i e_i + q_i \psi_1 \sum_{j=1}^n a_{ij} e_j + \psi_2 l_i \sum_{j=1}^n e_j - \gamma e_i^2 + s e_i$$

(50)

Extensive calculations are made in the appendices.

$^5$and that no distortion of optimal effort appears with taxation

$^6$see Guellec and Van Pottelsbergh de la Potterie (1997)
Proposition 14. We denote $\sigma = \frac{s}{4\gamma - 1}$, and $\mathbf{u}' = (I + \psi_1 \mathbf{A})$

1. When $(I_n - \phi_1 \mathbf{A})$ is invertible, the vector of quantities is:

$$\mathbf{q} = \frac{1}{1 - \rho} \left[ \mathbf{b}_\mu(G, \phi_1) - \rho \mathbf{q} \mathbf{b}_u(G, \phi_1) + \lambda \mathbf{b}_\nu(G, \phi_1) + \sigma \mathbf{b}_u'(G, \phi_1) \right]$$

Denoting $\mathbf{q}^*$ the vector of quantities without subsidy, this is equivalent to:

$$\mathbf{q} = \mathbf{q}^* + \frac{\sigma}{1 - \rho} \mathbf{b}_u'(G, \phi_1)$$

2. Equilibrium profits for firm $i$ are:

$$\pi_i = (1 - \frac{1}{4\gamma}) q_i^2 + \frac{\psi_1 \psi_2}{2\gamma} n c_i q_i + \frac{\psi_2}{2\gamma} [q + \psi_2 n_e - \psi_2/2] - \kappa l_i + \frac{1}{4\gamma} s^2 + l_i \psi_2 (n + 1 - \gamma) s$$

Corollary 15. The introduction of a subsidy distorts linearly a firm’s incentive $\Phi^e$ to move into the cluster by $\frac{\psi_2}{2\gamma} (n + 1 - \gamma) s$ compared to the case without subsidy.

Therefore, if this amount adding up to a firm’s previous incentive to enter makes its total incentive shift above $\kappa$, then the equilibrium will change to an equilibrium with one or several additional firms in the cluster (if the bonus is very large, several firms may see their incentive shift above $\kappa$). Note that, since this bonus linked to the subsidy policy does not include centrality measures, it will not change the process through which equilibria are determined.

7.3 Insights on the key-player policy

The idea of the key player policy in networks was first defined theoretically for crime network analysis by Ballester et al. (2006) (and first applied in Liu et al. (2012)), but was extended afterwards to R&D networks by König et al. (2014). What interests the policy maker is not so much the Nash equilibrium as the players on which he is likely to act efficiently. Therefore, the key player in a network is the node that participates the most to the output. In an R&D network, this node is the firm that, if removed, implies the highest drop in welfare. This is not necessarily the firm which procures the highest quantity, since it takes into account both the direct effect of the removal (drop in quantity), and the indirect effect of the removal (drop in spillovers received by other firms that lead to increased costs but reduced competition). The formal definition is that the key-player is the firm

$$i^* = \arg \max_{i \in N} c_i(G, \lambda_1), \text{ where } c_i \text{ is intercentrality, which is a quite complex notion defined in König et al. (2014), but which would calls for a formal definition when clusters are introduced in a single market.}$$

We then stick to intuitions on how a cluster policy may converge with a key player policy. Indeed, when a firm does not belong to the cluster, it means that it is not central enough, and therefore that removing it would have a relatively low impact both on quantities and on...
received spillovers. Thus, it appears that, since the threshold for an entry in the cluster is based on centrality, firms that enter are much more likely to be one of the key-players than peripheral firms. Moreover, that fact that they receive more spillovers from the cluster is defined in such a way in our model that it does not reduce their unit cost, so it only has a reduced effect on quantities, whereas it distorts their efforts and directly increases their profits. This means that introducing the cluster should increase the likeliness that the most central firms are also the key player, and in the end converge with a strict key player policy, since all the firms that participate the most to aggregate welfare end up being in the cluster. To our mind, this calls for further investigation on the potential convergence of these two types of policies.

8 Simulations of outcomes in a star network with 3 firms

We now propose to provide simulations of equilibrium outcomes in the second stage of the game (quantities and profits) for simple structures of networks and locational equilibria. An interesting network for simulations in the star network with three firms, because it allows to introduce heterogeneity between firms in a simple context:

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{fig1}
\caption{A star network with three firms}
\end{figure}

Since we have the structure of the network, we can assess, from the results we got in 3, the quantities produced (both at the individual and at the aggregate level). Moreover, by setting a location vector, we can derive the profits of each firm as well as aggregate welfare. Our goal here, rather than to address the locational equilibria as a function of $\kappa$, is to illustrate the comparative statics on the level of $\psi_2$. In these simulations, we consider that firms are homogenous in terms of costs ($\forall i, \mu_i = 4$). Note that we will only consider firm 2’s situation since firms 2 and 3 are homogenous.

8.1 Effects of locations and intensity of cluster spillovers on quantities and profits

8.1.1 Profits under high competition and low collaboration spillovers

One point that one can illustrate using simulations is the fact that increasing $\psi_2$ and having the most central firms enter the cluster may hurt the peripheral firms’ profits under
a high competition ($\theta$) and low spillovers from collaborations ($\psi_1$).

To exhibit this effect, we set $\psi_1 = 0.2$ (low spillovers from collaborations), $\gamma = 1$ (standard cost of effort), $\theta = 1$ (complete substitutability). We also remind that C1P denotes an equilibrium in which one firm only settles in the cluster. In this context, quantities of firm 2 evolve with $\psi_2$ as exposed in Figure 2. Under these parametric values, it is clear that less central firms are hurt by the introduction of the cluster when they are not able, themselves, to enter. This is showed in the blue line curves, where firm 2 is indifferent to the introduction of the cluster when $\psi_2$ is low enough. When $\psi_2$ increases, however, quantities of firm 2 clearly decrease. We also plotted total quantities since, as we previously showed, individual quantities decrease in total quantities, and the present decrease in firm 2’s quantities is clearly due to an increase in competition when firm 1 enters the cluster and when $\psi_2$ increases.

![Figure 2: Quantities produced by firm 2 and total quantities in function of $\psi_2$ in various locational equilibria](image)

7though no firms would pay $\kappa$ if $\psi_2$ is very low so these points are not relevant.
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Figure 3: Profits of firm 2 in function of $\psi_2$ in various locational equilibria with high competition and low $\psi_1$

Figure 4: Profits of firm 1 in function of $\psi_2$ in various locational equilibria with high competition and low $\psi_1$
Figure 3 clearly shows that this decrease in quantities in the C1P equilibrium (only firm 1 enters the cluster) is directly traduced in decreasing profits. It also exhibits that, even for a relatively low cost of entry in the cluster (here $\kappa = 2$), under such conditions of competition, an entry will be profitable for firm 2 only under a quite high $\psi_2$ (around 0.4) \(^8\).

Under the same parametric conditions, as soon as $\psi_2$ is above 0.4, we see in 4 that the introduction of a cluster is highly beneficial to the most central firm. The intersection of the green and blue curves shows the value of $\psi_2$ that implies a switch from a peripheral equilibrium to a C1P equilibrium when $\kappa = 2$. Moreover, we see that the profits of firm 1 increase when the two peripheral firms enter. This means that, even under fierce competition, the most central firm strictly benefits from the shift in effort made by the less central firms, and that spillovers widely offset the competition effect.

### 8.1.2 Profits with high collaboration spillovers

We can also compute profits for both firm 1 and firm 2 in the various locational equilibria, when firms gain more from collaborations ($\psi_1 = 0.5$) and under more “standard” competition conditions. As before, we choose $\forall i, \mu_i = 4, \gamma = 1, \kappa = 2$, but we now set $\theta = 0.7$ to get a lower competition.

---

\(^8\) when firm 1 entered, but firm 2 cannot enter if firm 1 has not. We also note that this threshold decreases quite slowly when $\psi_1$ increases.
In such a setting, figures 5 and 6 show the respective profits of firm 1 and firm 2 as functions of $\psi_2$. We first see that firm 1’s profits are always above firm 2’s. Moreover, in such conditions, we see that even with a relatively high substitutability of goods ($\theta = 0.7$), the spillover effect offset by far the competition effect, and firm 2’s profits always increase when $\psi_2$ increase in the C1P equilibrium, i.e. when itself locates in the periphery.

### 8.2 Welfare analysis in the 3 firms star network

The main issue that a policy maker wants to address is the global effect on welfare that a cluster policy will induce. In the previous analysis, we saw that it was very difficult to conclude on the sign of the overall effect of financing additional entries in the cluster or of an increase in $\psi_1$ since the additional spillover effects are opposed to competition effects and to the costs incurred in the policy design. Therefore, we provide simulations on welfare, keeping our previous set of parameters, except from the level of collaborations spillovers which we set at a more reasonable level: $\psi_1 = 0.3$ (reminder: $\mu = 4$, $\gamma = 1$, $\kappa = 2$, $\theta = 0.7$). We follow our previous assumption that the representative consumer is taxed to finance the increase in $\psi_2$. The cost of increasing $\psi_2$ is determined by a convex function $\chi$, which we set:

$$\chi : \psi_2 \to 12.\psi_2^2 + 0.2$$

This means that we here consider a “natural level” of cluster spillovers of $\psi_2$, level that can be increased by the policy maker with convex costs. Under such conditions, we can plot utility of the representative consumer and welfare in this economy as functions of $\psi_2$. 

Figure 6: Profits of firm 2 in function of $\psi_2$ in various locational equilibria with high $\psi_1$
in various locational equilibria. We dropped the peripheral equilibrium because it yields constant utility and welfare over $\psi_2$.

Figure 7: Aggregate utility in function of $\psi_2$ in central and intermediate equilibria

Figure 8: Aggregate welfare in function of $\psi_2$ in central and intermediate equilibria

Figure 7 clearly shows that, under these parameters, there exists a value of $\psi_2$ which maximizes aggregate utility for each locational equilibrium. However, we see that in the

9Note that, if we sometimes spoke of $\psi_2$ as a rate, the expression is improper and there is no reason why the range of $\psi_2$ should be bounded by 1, so we consider a range a little wider.
C1P equilibrium, gains in terms of utility are very small, and that \( \psi_2 \) can be set almost indifferently for consumers between 0.2 and 0.6. As soon as the cost lies exclusively on consumers, a policy favorable to an increase in \( \psi_2 \) is only widely beneficial to consumers when a large number of firms settle in the cluster.

Figure 8 exhibits the evolution of aggregate welfare with \( \psi_2 \) in this economy. A first point to notice is that the maxima of welfare and of utility are not reached for the same value of \( \psi_2 \). Of course, this would change if we were to change the structure of financement of cluster policies\(^{10}\). Moreover, we can state that different maxima are reached when the locational equilibrium varies. This is of course dynamic: if the government chooses at the same time to introduce a cluster policy and to increase \( \psi_2 \), some firms will have an incentive to enter, so the locational equilibrium will change and the welfare path will be different. Under our parametric conditions, we see that for relevant values of \( \psi_2 \) (over 0.4, since even the most central firm does not enter the cluster under this value), the central equilibrium is always welfare improving compared with the C1P equilibrium. This confirms our intuition that the conditions given in section 5 for an entry to be welfare improving are not restrictive, and that the policy maker will, in many situations, have an incentive to lead policies that both incentivize firms to enter the cluster and increase the intensity of cluster spillovers.

\[ \text{9 Conclusion} \]

We built a model based on a 2-stage game, introducing a double dimension of spillovers. One dimension consisted of R&D collaborations between firms that chose to share their research with others to decrease production costs of both partners. This allowed us to base the analysis on the network of these collaborations. The second dimension consisted of localized spillovers, through the introduction of an industrial cluster, in which firms could choose to enter or not in the first stage of the game, and get a bonus profit proportional to the aggregate efforts in their market. The interpretation we made of this bonus was that firms were able, through their location in the cluster, to be closer to innovation processes at the market level, and therefore improved their position as innovators in the market. This localized dimension of spillovers was widely inspired by the extensive empirical literature on the localization of spillovers.

We first solved the second stage of our model, for which we showed that efforts and quantities were increasing in firm's centrality. We derived the unique Nash equilibrium quantities and aggregate quantities as functions of the network structure, location structure and parameters. We endogenized the location choice of the firms in a first stage of the game and showed that there was a unique equilibrium for each value of the cost to enter in the cluster, and that the most central firms in the network were always the first to enter. We proposed

\(^{10}\)Here it was simpler to assume that consumers incurred the cost, otherwise we would have needed to introduce taxation on consumers and on firms, which would have distorted the behavior of the latter. However, we believe that such a mixed taxation would be an interesting extension for this model.
some comparative statics on the first stage of the game, addressing what would happen in the market if an additional firm was to enter in the cluster. We derived this effect on quantities, profits and utility, through constraints on the parameters of the model. Afterwards, we addressed the issue of the stability of the network in a given locational equilibrium, and found that the stable network was likely to be very assortative.

In the end, we analysed different types of potential public policies in this context, based on two instruments: increasing the intensity of cluster spillovers through investment in research facilities, financing innovation with R&D subsidies. We supplemented this with insights on key-player policies. We then proposed simulations of the outcomes of the model in a star network with three firms, mainly focusing on the effects of an increase in the intensity of cluster spillovers.

We relied a lot on recent works from the social networks literature, mainly Helsley and Zenou (2013), that introduced location choice in a social networks framework, and König et al. (2014), that covered R&D collaborations in a very extensive and thorough way. To the best of our knowledge, this is the first attempt to introduce the two dimensions of spillovers we named in a single model, however both dimensions have been covered with multiple empirical applications. As a first attempt, it of course has strong disadvantages, among which heavy computations that sometimes make interpretations quite difficult (especially in the comparative statics part). Nevertheless, it seemed interesting to focus on the microfoundation of such a discussed topic, both in the empirical and macro literature, and in public policy evaluations of innovation support. On top of this, we believe that a framework with at the same time oligopolistic competition, social networks and location choices was quite realistic and allowed to draw intuitive conclusions for public polices, even though the resolution of the model was heavier and less intuitive, and that further investigations should be dedicated to the public policy analysis.

There are several extensions one could propose for this model. The first one is of course to try to estimate this model empirically. The R&D collaborations are recorded in the MERIT-CATI dataset. As for cluster data, a way to estimate simply the data would be to select a single cluster that is supposed to be very innovative (as, for instance, will the “plateau de Saclay” project be in France), and to draw a network with the firms around this cluster. This would however require to extend our model to several markets, with a competition matrix, as König et al. (2014) introduced. Another extension would be to provide extensive simulations for more complex network structures with some very central firms and some hardly connected firms, that should be harmed by the introduction of a cluster even without an extremely high competition. Finally, providing a process of network formation as well as efficiency of the networks would greatly improve the model.
Appendices

A) Precisions on the derivation of equilibrium profits conditional on locations

Written extensively, profits write:

$$
\pi_i = (\alpha - \bar{c}_i)q_i - g_i \sum_{j \neq i} q_j + q_i \left( \frac{q_i + \psi_2 l_i}{2\gamma} + \frac{q_i \psi_1}{2\gamma} \sum_j a_{ij} q_j + q_i \psi_1 \psi_2 \sum_j a_{ij} l_j + \psi_2 l_i \sum_j \frac{q_j + \psi_2 l_j}{2\gamma} \right) \\
- \gamma \left( \frac{q_i + \psi_2 l_i}{2\gamma} \right)^2 - \kappa l_i \tag{54}
$$

We recombine to gather the terms that do not involve $$\psi_2$$ or $$l_i$$:

$$
\pi_i = (\alpha - \bar{c}_i)q_i - g_i \sum_{j \neq i} q_j + \frac{q_i \psi_1}{2\gamma} \sum_j a_{ij} q_j + q_i \left( \frac{1}{4\gamma} q_i^2 - (1 - \frac{1}{4\gamma}) q_i^2 \right) \\
+ \frac{\psi_1}{2\gamma} q_i l_i + q_i \frac{\psi_1 \psi_2}{2\gamma} \sum_j a_{ij} l_j + \psi_2 l_i \sum_j \frac{q_j + \psi_2 l_j}{2\gamma} - \gamma (2q_i l_i + q_i^2) \tag{55}
$$

Following the demonstration led by König et al. (2014) page 62, the standard terms recombine in: $$(1 - \frac{1}{4\gamma})q_i^2$$.

B) Conditions on $$\kappa$$ for locational equilibria

→ For the central equilibrium to hold we need the incentive for the least central player, $$n$$, to be higher than the cost, i.e.

$$\Phi^C(m_{nn}) > \kappa \tag{56}$$

→ For the equilibrium where all players but $$n$$ locate in the cluster to hold we need $$\Phi^{CP1}(m_{n-1,n-1}) > \kappa$$ and $$\Phi^C(m_{nn}) < \kappa$$.

What is going to change in the incentive for a firm to enter is the “non self-loops” part of the incentive, thus the $$r_i$$ we defined earlier (the part of the expression of $$q_i$$s only composed of self-loops). Thus, we note that:

$$r_{n-1}^C = m_{n-1,n} \lambda_2 + r_{n-1}^{CP1} = \phi_2 m_{n-1,n} + r_{n-1}^{CP1}$$

This implies:

$$\Phi^C(m_{n-1,n-1}) - \Phi^{CP1}(m_{n-1,n-1}) = \frac{\lambda_2 \psi_2}{2\gamma (1 - \rho)} \left[ \psi_3 a_{n,n-1} \left( \frac{m_{n-1,n-1}^2}{1 - \rho} + m_{n-1,n-1} + m_{n-1,n} \right) + \frac{m_{n-1,n}}{1 - \rho} \right] \tag{57}$$
We also have: \( \Phi^{CP1}(m_{n,n}) = \Phi^C(m_{n,n}) \).

The condition for CP1 to hold can then be expressed as:

\[
\Phi^C(m_{n,n}) < \kappa \leq \Phi^C(m_{n-1,n-1}) - \frac{\lambda_2 \psi_2}{2\gamma(1-\rho)} \left[ \psi_1a_{n,n-1}\left( \frac{m_{n-1,n-1}^2}{1-\rho} + m_{n-1,n-1} + m_{n-1,n} \right) \right. \\
\left. + \frac{m_{n-1,n}}{1-\rho} \right] (58)
\]

\( \rightarrow \) Using the same method as before, the condition for CP2 to hold is:

\[
\Phi^C(m_{n-1,n-1}) - \frac{\lambda_2 \psi_2}{2\gamma(1-\rho)} \left[ \psi_1a_{n,n-1}\left( \frac{m_{n-1,n-1}^2}{1-\rho} + m_{n-1,n-1} + m_{n-1,n} \right) \right. \\
\left. + \frac{m_{n-1,n}}{1-\rho} \right] < \kappa \\
\Phi^C(m_{n-2,n-2}) - \frac{\lambda_2 \psi_2}{2\gamma(1-\rho)} \left[ \psi_1a_{n-1,n-2}\left( \frac{m_{n-2,n-2}^2}{1-\rho} + m_{n-2,n-2} + m_{n-2,n-1} \right) \right. \\
\left. + \frac{m_{n-2,n-1}}{1-\rho} \right] (59)
\]

\( \rightarrow \) Iterating with the same method to the most central one . . .

\( \rightarrow \) The condition for a peripheral equilibrium to hold is:

\[
\Phi^C(m_{11}) - \frac{\lambda_2 \psi_2}{2\gamma(1-\rho)} \sum_{j=2}^{n} \left[ \psi_1a_{j,j-1}\left( \frac{m_{j-1,j-1}^2}{1-\rho} + m_{j-1,j-1} + m_{j-1,j} \right) \right. \\
\left. + \frac{m_{j-1,j}}{1-\rho} \right] < \kappa (60)
\]

C) Derivation of equilibrium with R&D subsidy

We introduce a direct linear subsidy on R&D efforts, that changes the profits structure to:

\[
\pi_i = (\alpha - \bar{c}_i)q_i - q_i^2 - \theta \sum_{j \neq i} q_iq_j + q_ie_i + q_i\psi_1 \sum_{j=1}^{n} a_{ij}e_j + \psi_2l_i \sum_{j=1}^{n} e_j - \gamma e_i^2 + se_i (61)
\]

Therefore, the FOC with respect to effort is:

\[
\frac{\partial \pi_i}{\partial e_i} = q_i - 2\gamma e_i + \psi_2l_i + s = 0
\]

i.e.

\[
e_i = \frac{1}{2\gamma}(q_i + s + \psi_2l_i)
\]
And the FOC with respect to quantity is:

\[
\frac{\partial \pi_i}{\partial q_i} = \alpha - \bar{c}_i - 2q_i - \theta \sum_{j \neq i} e_i + \psi_1 \sum_{i=1}^{n} a_{ij}e_j = 0
\]

\[ q_i(2 - \frac{1}{2\gamma}) = \alpha - \bar{c}_i - 2q_i - \theta \sum_{j \neq i} q_j + \frac{1}{2\gamma} (s + \psi_2 l_i) + \psi_1 \sum_{i=1}^{n} a_{ij}(q_j + s + \psi_2 l_j) \]

Denoting \( \sigma = \frac{s}{4\gamma - 1} \) and \( d_i \) the degree of firm \( i \),

\[ q_i(1 - \rho) = \mu_i - \rho \bar{q} + \lambda \sum_{j=1}^{n} a_{ij}q_j + \lambda_2 l^i + \sigma (1 + \psi_1 d_i) \]

\[ q(I_n - \phi_1 A) = \frac{1}{1 - \rho} \left[ \mu - \rho \bar{q} u + \lambda_2 l' + \sigma [(I_n + \psi_1 A) u] \right] \]

When \((I_n - \phi_1 A)\) is invertible, we therefore have the vector of quantities:

\[ q = \frac{1}{1 - \rho} \left[ b_{\mu}(G, \phi_1) - \rho \bar{q} b_u(G, \phi_1) + \lambda_2 b_{l'}(G, \phi_1) + \sigma b_{u^e}(G, \phi_1) \right] \]

To derive profits, we proceed as in the case without subsidy, and simplify "normal terms" according to König et al. (2014). Therefore, using (14),

\[ \pi_i = (1 - \frac{1}{4\gamma}) q_i^2 + \left[ \frac{\psi_2}{2\gamma} [\bar{q} + \psi_2 n_c - \frac{\psi_2}{2}] - \kappa \right] l_i + s \frac{q_i + \psi_2 l_i + s}{2\gamma} + \frac{\psi_2 l_i}{2\gamma} \sum_{j=1}^{n} s - \gamma \frac{2\psi_2 l_iss}{4\gamma^2} \]

i.e.

\[ \pi_i = (1 - \frac{1}{4\gamma}) q_i^2 + \frac{\psi_1 \psi_2}{2\gamma} n_c q_i + \left[ \frac{\psi_2}{2\gamma} [\bar{q} + \psi_2 n_c - \frac{\psi_2}{2}] - \kappa \right] l_i + \frac{1}{4\gamma} s^2 + l_i \frac{\psi_2}{2\gamma} (n + 1 - \gamma) s \]

References


Jackson, M. O. and Y. Zenou (2012). Games on networks. CEPR.


