Appendix A: ALL YOU WANTED TO KNOW ABOUT Dixit-Stiglitz BUT WERE AFRAID TO ASK

CES preferences are at the heart of the Dixit-Stiglitz monopolistic competition model (Dixit and Stiglitz 1977). They can be expressed in terms of discrete varieties or a continuum of varieties. To wit:

\[ U = C_M; \quad C_M = \left( \int \sum_{i=0}^{N} c_i^{1-\sigma} \, di \right)^{1/1-\sigma}, \quad C_M' = \left( \sum_{i=1}^{N} c_i^{1-\sigma} \right)^{1/1-\sigma}, \quad \sigma > 1 \]

where \( N \) is the mass of varieties in the former (continuum version) and the number of varieties in the latter (discrete version). The corresponding indirect utility functions of both versions can be written as \( E/P_M \) where the price indices are, respectively:

\[ V = \frac{E}{P_M}; \quad P_M = \left( \int p_i^{\alpha} \, di \right)^{1/1-\sigma}, \quad P_M' = \left( \sum_{i=1}^{N} p_i^{\alpha} \right)^{1/1-\sigma}; \quad \sigma > 1; \]

\( P_M \) is called a 'perfect' price index since it translates expenditure, \( E \), into utility; this is useful in many situations since it equates real income with utility.

The first order condition of utility maximisation is (with discrete varieties):

\[ C_M \left( \sum_{i=0}^{N} c_i^{1-\sigma} \right)^{-1} c_j^{1-\sigma} = \lambda p_j, \quad \forall j \]

Multiplying both sides by \( c_j \), summing across varieties and using the budget constraint, \( E=\Sigma_i p_i c_i \), yields one expression for the Lagrangian multiplier, namely \( \lambda=C_M/E \). Alternatively, isolating \( c_j \) on the left-hand side, multiplying both sides by \( p_j \), summing across varieties and using the budget constraint, \( \lambda \) can be show to be equal to \( (\Sigma_i p_i^{\alpha})^{-1}(E^{1-\sigma})(C_X/(\Sigma_i c_i^{1-\sigma})) \). Plugging the first or second expression for \( \lambda \) into the first-order condition yields the inverse or direct demand curves, respectively. These are:

\[ p_j = \frac{c_j^{1-\sigma}}{\sum_{i=1}^{N} c_i^{1-\sigma}} E, \quad c_j = \frac{p_j^{\sigma}}{\sum_{i=1}^{N} p_i^{\sigma}} E, \quad \forall j \]

It is convenient to use the indirect demand function when assuming quantity (Cournot) competition and to use the direct demand function when working with price (Bertrand) competition (as we shall see, these two forms of competition lead to the same behaviour when firms are atomistic). Derivation of demand functions for the continuum version is identical except of course the summations are replaced by integrals.

The CES utility function is often referred to as “love of variety” preferences. To understand why, we show that the same level of expenditure spread over more varieties increases utility. If all varieties are priced at \( 'p' \), consumption of a typical variety is \( E/Np \). Substituting this into the utility function implies that \( U=N^{1/(1-\sigma)}(E/p) \) and we see that utility rises with \( N \), so in this sense, consumers love variety for...
variety’s sake. Moreover, even if each variety is priced differently, adding a new one increases utility if prices of the existing varieties are unchanged; this is very easily seen by using the expression for the perfect price index in the discrete case.

A.1 Dixit-Stiglitz Competition, Mill Pricing and Firms’ First Order Conditions

Dixit-Stiglitz monopolistic competition is highly tractable since a firm’s optimal price is a constant mark-up over marginal cost. This is unusual since in most forms of imperfect competition, the optimal price-marginal cost mark-up depends upon the degree of competition, with the mark-up increasing as the degree of competition falls. For example, the optimal mark-up often depends upon the firm’s market share, but since the market share depends upon prices, one needs to solve all firms’ first order conditions simultaneously. If additionally the number of firms is determined by free entry, finding equilibrium prices can require the simultaneous solution of many equations, some which will be non-linear. Fixed mark-ups permit us to avoid all this.

To get started, consider Cournot competition among N firms in a single market, with each firm producing a symmetric variety subject to a homothetic cost function. The typical firm’s objective function is revenue, \( p_c j c_j \), minus costs, \( (a_m c_j + F)w \), where \( wF \) is the fixed cost, \( wa_m c \) is the variable cost and \( w \) is the wage.

Using the discrete-varieties version of the indirect demand function, the Cournot first order condition is:

\[
(A-1) \quad p(1 - \frac{1}{\varepsilon}) = wa_M; \quad \frac{1}{\varepsilon_{Cournot}} = \frac{1}{\sigma} + (1 - \frac{1}{\sigma})s, \quad \varepsilon_{Bertrand} = \sigma - (\sigma - 1)s
\]

where “s” is the market share of the typical variety; with symmetry \( s=1/N \). Using the direct demand function and price competition yields the third expression. Note that under both conjectures, the perceived elasticity, \( \varepsilon \), falls as ‘s’ rises.

Note that as long as ‘s’ is not zero, the degree of competition does affect the mark-up and thus pricing behaviour. For example, with symmetry and Cournot competition, the equilibrium mark-up is \( (1-1/\sigma)(1-1/N) \), so the equilibrium mark-up falls as the number of competitors rises. This is called ‘small-group’ monopolistic competition. An interesting extreme case – which is at the root of the Dixit-Stiglitz monopolistic competition – is where \( N \) rises to infinity and the perceived elasticity \( \varepsilon \) equals \( \sigma \) (under both Cournot and Bertrand competition). Since the perceived elasticity is invariant to \( N \), the mark-up is constant. This extreme case is what Chamberlain called the “large-group” case and it is what Dixit-Stiglitz monopolistic competition assumes. Four comments are in order.

First, note that with an infinite number of atomistic competitors – i.e. under Dixit-Stiglitz assumptions – equilibrium pricing does not depend upon the type of competition. Bertrand and Cournot competition produce the same result. While this is convenient, it is a strong assumption that rules out many interesting effects, such as the pro-competitive effect.

Second, in the discrete varieties version of Dixit-Stiglitz preferences (assumed in the original 1977 article), one must assume that \( N \) is large enough to approximate \( \varepsilon \) with \( \sigma \). With the continuum of varieties case, there are an uncountable infinity of varieties, so “s” is automatically zero.
Third, the invariance of the Dixit-Stiglitz mark-up to changes in the number (mass) of firms is easily understood. One starts with the assumption that the number of competitors is infinite, so adding in more competitors has no effect. Infinity, after all, is a concept, not a number.

Fourth, the invariance of the mark-up implies that so-called mill pricing is optimal for firms. That is, with iceberg costs, if it costs $T_1$ to ship the goods to market 1, and $T_2$ to ship them to market 2, firms will fully pass the shipping costs on to consumer prices, so the ratio of consumer prices in market 1 to market 2 will be $T_1/T_2$. This is called mill pricing, or factory gate pricing, since it is as if the firm charged the same price “at the mill” or at the factory gate, with all shipping charges being born by consumers. Another way of saying this is that with mill pricing, a firm’s producer price is the same for sales to all markets.

### A.2 Operating Profits, Free Entry and the Invariance of Firm Scale

One extremely handy, but not very realistic, aspects of Dixit-Stiglitz monopolistic competition is the invariance of equilibrium firm scale. This is a direct and inevitable implication of the constant mark-up, free entry and the homothetic cost function.

Plainly, a fixed mark-up of price over marginal cost implies a fixed operating profit margin. It is not surprising, therefore, that there is a unique level of sales that allows the typical firm to just break even, i.e. to earn a level of operating profit sufficient to cover fixed costs. The first order condition (the pricing equation) can be arranged as:

$$
 p(1 - \frac{1}{\sigma}) = w_{aM} \Rightarrow (p - w_{aM}) = \frac{p}{\sigma} \Rightarrow (p - w_{aM})c = \frac{pc}{\sigma} \Rightarrow (p - w_{aM})c = \frac{w_{aM}c}{\sigma - 1}
$$

The second to last expression shows that operating profit, $(p-w_{aM})c$, equals an invariant profit margin (namely, $1/\sigma$) times the value of consumption at consumer prices. The last equation is derived using the formula for the equilibrium price. The constancy of equilibrium firm scale, i.e. the volume of sales/production necessary for a typical firm to break even is obvious when the scale economies take the familiar form of a linear cost function, namely $w(F+a_{MC})$, where $wF$ is the fixed cost and $w_{aM}c$ is the variable cost. The zero profit condition in this case is just:

$$
 \frac{w_{aM}c}{\sigma - 1} = wF \Rightarrow c = \frac{F(\sigma - 1)}{a_{MC}}
$$

Observe that equilibrium firm scale depends on two cost parameters, $F$ and $a_{MC}$, and a demand parameter, $\sigma$.

As it turns out, the invariance of equilibrium scale economies demonstrated above is quite a general proposition, at least for one common measure of scale, viz. the scale elasticity. As long as the price-marginal cost mark-up is fixed and the zero profit condition holds, the scale elasticity, i.e. $\chi=(dC/dx)(x/C)$, where $x$ is firm output/sales and $C$ is the cost function, must be constant. To see this, note that with zero profit, price must equal average cost, so the first order condition can be written as $AC/MC=(1-1/\sigma)$, where MC and AC are marginal and average cost respectively. But, $(dC/dx)(x/C)$ is just $MC/AC$, so $1/\chi=(1-1/\sigma)$. 

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Note that, the scale elasticity is a measure that has its limitations. For instance, if the cost function is not homothetic in factor prices, a given scale elasticity does not coincide one-to-one with firm size. For instance, if capital is used only in the fixed cost and labour only in the variable costs, then the scale elasticity is \((-t/(w_m x+1))^{-1}\). Even if this is constant at, say, \(-1/\sigma\), the firm size that corresponds to this depends upon the \(t/w\) ratio. Since trade costs can in general affect factor prices, this means that trade costs can also affect firm scale, even in the Dixit-Stiglitz model.

### A.3 Invariance of Firm Scale with Trade

The simplicity that comes with Dixit-Stiglitz monopolistic competition is especially apparent when dealing with multiple markets. In particular, when we assume that trade costs are “iceberg” in nature (i.e. are proportional to marginal production costs since a fraction of shipped goods disappear in transit), solving the multi-market problem is no more difficult than solve the single market problem.

To understand the source of the simplifications, we start with a more general set of assumptions. Suppose there are two markets, local and export, and that it costs \(T^*\) to ship one unit of the good to the export market and \(T\) to ship it to the local market. These costs are not of the iceberg type.

A typical firm has \(p(1-1/\sigma)=(w_m + T)\) and \(p^*(1-1/\sigma)=(w_m + T^*)\) as its first order conditions, where \(p^*\) is the consumer price in the distant market. Rearranging these conditions shows that operating profit – which we denote at \(\pi\) – is proportional to the value of retail sales, \(R\). Specifically, \(\pi=R/\sigma\), so the free entry condition requires that \(R/\sigma=w_F\) as in the single market case without trade costs. However, now \(R=pc+p^*c^*\), where \(c\) and \(c^*\) are consumption in the local and export markets. Rearranging, we have that \(c+c^*=w_F/\sigma-\psi c^*\), where \(1+\psi=p^*/p\) and from the first order conditions, \(p^*/p\) equals \((w_m + T^*)/(w_m + T)\). The left-hand side is clearly not constant because the right-hand side is not. Indeed, in general both terms on the right-hand side may vary.

What is needed to make \(c+c^*\) invariant to trade costs? If the cost function is homogenous, the fixed costs \(w_F\) is proportional to the price (recall that price is proportional to marginal cost), so \(w_F/\sigma/p\) will not vary with relative factor prices. Nevertheless, scale will vary since both \(\psi\) and \(c^*\) vary with trade costs. To make \(\psi c^*\) constant, we assume that trade costs are “iceberg” in the sense that a certain fraction of each shipment disappears in transit. This makes trade costs proportional to marginal cost. For example, if marginal costs are \(w_m(1+t^*)\) for export sales and \(w_m(1+t)\) for local sales, then \(\psi\) equals \((t^*-t)/(1+t)\). In this case, we can without further loss of generality absorb \(1+t\) into the definition of \(a_k\) and define trade costs as zero for local sales and \(t”=t^*-t\) for distant sales (this is standard practice). Moreover, with iceberg costs we have that \((1+t”)c^*\) equals the quantity produced for the distant market since the quantity produced and shipped is always \(1+t”\) times consumption.

In summary, the invariance of firm size (as measured by production) to trade costs is a result that is very sensitive to special assumptions and functional forms. Trade costs must be iceberg, the cost function must be homothetic and equilibrium prices must be proportional to marginal costs (this in turn requires mill pricing).