1 A Cash-in-advance Model

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In this problem set, we will try to answer some questional issues on money and business cycle fluctuations. More precisely, we will follow Cooley and Hansen [T.F. Cooley and G.D. Hansen (1989) “The Inflation Tax in a Real Business Cycle Model”, The American Economic Review, 79(4), pp.733–48] and answer the following questions:

- Does money and the form of money supply rules affect the nature and amplitude of the business cycle?
- How does anticipated inflation affect the nature and amplitude of the business cycle?

The model

Let’s consider an economy populated by a representative household seeking to maximize an intertemporal utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t (\ln c_t - Bh_t), \ 0 < \beta < 1$$

depending on consumption $c_t$ and labour effort $h_t$, and where $E_0$ is the expectation operator with respect to information available at date 0.

The model - cont’d

This household faces a budget constraint:

$$c_t + x_t + m_t/p_t \leq w_t h_t + r_t k_t + (m_{t-1} + (g_t - 1)M_{t-1})/p_t$$

where $x_t$ is the investment, defined by $k_{t+1} = (1-\delta)k_t + x_t$, $0 \leq \delta \leq 1$, $m_t$ the nominal money balances, $p_t$ a price index, $w_t$ the real wage, $r_t$ the real interest rate and $k_t$ the capital stock ($k_0$ given.) Per capita money supply, $M_t$, evolves according to $M_t = g_t M_{t-1}$, with $g_{t+1} = g_t \bar{g}^{1-\alpha} \exp(\xi_{t+1})$, $\xi_t$ iid with variance $\sigma_{\xi}^2$. The household faces another constraint on cash good

$$p_t c_t \leq m_{t-1} + (g_t - 1)M_{t-1}$$

Question 1

Give a rationale for a cash-in-advance constraint. What is the basic difference with the money-in-utility-function paradigm?

The cash-in-advance paradigm

There are two main timing conventions

- Lucas: Households visit the financial market after seeing current period shocks but before purchasing cash. This means households can adjust their holdings of cash in the financial market to exactly meet their consumption needs.
- Svensson: Goods market opens before the asset market. Households have to purchase goods using cash they carried over from the previous period. Holdings of money are chosen before households see the current period shock. There is a precautionary motive for holding cash.
CIA vs MIU models

The main difference

- MIU: households gain utility flows from their money holdings, no matter how they use cash.
- CIA: give a description of what people do with their money holdings.

Question 2

Write the lagrangian for the household problem and derive the first order conditions for an optimal path.

The household wants to maximize its intertemporal utility using \( c_t \), \( h_t \), \( m_t \) and \( x_t \) as controls. The Lagrangian of that problem is thus:

\[
L = E_0 \sum_{t=0}^{\infty} \beta^t \left[ \ln c_t - Bh_t 
+ \lambda_t \left( w_t h_t + r_t k_t + \frac{m_{t-1} + (g_t - 1)M_{t-1}}{p_t} \right)
- c_t - k_{t+1} + (1 - \delta)k_t - \frac{m_t}{p_t} \right]
+ \mu_t \left( \frac{m_{t-1} - (g_t - 1)M_{t-1}}{p_t} - c_t \right)
\]

The first-order conditions with respect to \( c_t \), \( h_t \), \( m_t \) and \( k_t \) are then respectively (with the expectation operator implicitly appended):

\[
\begin{align*}
(c_t) \quad & \frac{1}{c_t} = \lambda_t + \mu_t \\
(h_t) \quad & \frac{w_t}{B} = \lambda_t \\
(m_t) \quad & E_t \left\{ \frac{p_{t+1}}{p_t} \beta (\lambda_{t+1} + \mu_{t+1}) \right\} = \lambda_t \\
(k_t) \quad & E_t \left\{ \frac{\beta \lambda_{t+1}}{\lambda_t} (r_{t+1} + 1 - \delta) \right\} = 1
\end{align*}
\]

The first relation equals the cost of holding one more unit of money (as a loss in consumption) to its future benefits (larger future consumption and loosened cash constraint). The second holds the trade-off between labour and leisure. The third is a asset pricing equation, expression the value of the liquidity services of money. The last relation is the Euler relation.

Question 3

In which variables one can found nominal trend? Propose a method to make the problem stationary.

There is a nominal trend in the optimal level of cash holdings. The problem can be made stationary by defining:

\[ \hat{m}_t = \frac{m_t}{M_t} \text{ and } \hat{p}_t = \frac{p_t}{M_t} \]

Question 4

The firm in the economy produces output according to a Cobb-Douglas production function

\[ y_t = \exp(z_t)k_t^{\theta}h_t^{1-\theta}, \quad 0 \leq \theta \leq 1, \text{ with } z_{t+1} = \gamma z_t + \varepsilon_{t+1}, \quad 0 \leq \gamma \leq 1 \text{ and } \varepsilon_t \text{ iid with mean 0 and variance } \sigma^2 \]

Give the first order conditions for the firm’s problem.
Optimal level of production input are given by

\[ w_t = (1 - \theta) \frac{y_t}{h_t} \]

and

\[ r_t = \theta \frac{y_t}{k_t} \]

**Question 5**
Compute the steady state for this economy.

**Steady state**
At the steady-state, we have

\[ g = \bar{g} \]
\[ \bar{z} = 0 \]
\[ \bar{m} = 1 \]
\[ \bar{r} = \frac{1}{\beta} - 1 + \delta \]

**Steady state - cont’d**
Then, according to first order conditions, constraints, and functional forms

\[ \bar{w} = (1 - \theta)(\theta/\bar{r})^{(1/1-\theta)} \]
\[ \bar{\lambda} = B/\bar{w} \]
\[ \bar{\epsilon} = \beta/(\bar{\lambda}\bar{g}) \]
\[ \bar{\rho} = 1/\bar{\epsilon} \]
\[ \bar{h} = \bar{\epsilon}/(\bar{w} + (\theta/\bar{r})^{1/(1-\theta)}(\bar{r} - \delta)) \]
\[ \bar{g} = \bar{w}\bar{h}/(1 - \theta) \]
\[ \bar{k} = \theta\bar{g}/\bar{r} \]
\[ \bar{x} = \delta\bar{k} \]
\[ \bar{\mu} = 1/\bar{\epsilon} - \bar{\lambda} \]

**Simulations**
We use the following calibration

\[ \begin{array}{cccccccccc}
\beta & \theta & \delta & B & \gamma & \sigma_r & \bar{g} & \alpha & \sigma_\bar{g} \\
0.99 & 0.36 & 0.025 & 2.86 & 0.95 & 0.00721 & 1.015 & 0.48 & 0.009
\end{array} \]

**Question 6**
Simulate the model with 50 draws of 115 periods long. After logging and detrending, compute the standard deviation and correlation with output for the following variables: output, consumption, investment, capital stock, hours, productivity, and price level.

The results of the simulation are:

**Question 7**
Redo the simulations with \( \sigma_\bar{g} = 0 \). Compare these results with the Hansen(1985)’s indivisible labor model (see below) and conclude on the initial questions.
Table 1: Simulation 1, 50 draws, 115 periods

<table>
<thead>
<tr>
<th>Series</th>
<th>Std Dev.</th>
<th>Corr w. Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>1.79(0.19)</td>
<td>1.00(0.00)</td>
</tr>
<tr>
<td>Consumption</td>
<td>0.66(0.07)</td>
<td>0.70(0.05)</td>
</tr>
<tr>
<td>Investment</td>
<td>5.88(0.58)</td>
<td>0.97(0.01)</td>
</tr>
<tr>
<td>Capital Stock</td>
<td>0.50(0.09)</td>
<td>0.07(0.05)</td>
</tr>
<tr>
<td>Hours</td>
<td>1.36(0.14)</td>
<td>0.98(0.00)</td>
</tr>
<tr>
<td>Productivity</td>
<td>0.53(0.07)</td>
<td>0.87(0.03)</td>
</tr>
<tr>
<td>Price level</td>
<td>1.94(0.27)</td>
<td>-0.26(0.17)</td>
</tr>
</tbody>
</table>

Table 2: Simulation 2, 50 draws, 115 periods, $\sigma_\xi = 0$

With $\sigma_\xi = 0$, the results are: According to these simulation, money does affect the business cycle through the cash-in-advance constraint. In particular, consumption and price level are more volatile when money supply is stochastic, but consumption is less correlated with output.