Optimal Sovereign Default*

Klaus Adam, University of Mannheim and CEPR
Michael Grill, Deutsche Bundesbank

August 29, 2012

Abstract

When is it optimal for a government to default on its legal repayment obligations? We answer this question for a small open economy with domestic production risk in which the government finances itself by (optimally) issuing non-contingent debt. We show that it is Ramsey optimal to occasionally deviate from the legal repayment obligation and to repay debt only partially, even if such deviations give rise to significant ‘default costs’. Optimal default improves the international diversification of domestic output risk, increases the efficiency of domestic investment and - for a wide range of default costs - significantly increase welfare relative to a situation where default is simply ruled from Ramsey optimal plans. We analytically show that default is optimal following adverse shocks to domestic output, especially for very negative international wealth positions. A quantitative analysis reveals that default is optimal only in response to disaster-like shocks to domestic output, or when small adverse shocks push international debt levels sufficiently close to the country’s borrowing limits.

JEL Class. No.: E62, F34

1 Introduction

When is it optimal for a sovereign to default on its outstanding debt? We analyze this hotly debated question in a quantitative equilibrium framework in which a country can internationally borrow and invest to smooth out the consumption implications of domestic productivity shocks. Importantly, we determine the default policies that maximize the country’s ex-ante welfare, i.e., derive the Ramsey optimal policy under full commitment. We show that Ramsey policies will involve occasional sovereign

*Thanks go to seminar participants at CREI Barcelona, London Business School, Bank of Portugal, the 2011 Bundesbank Spring Conference, and to Fernando Broner, Jordi Galfí, Pierre Olivier Gourinchas, Jonathan Heathcote, Felix Kuebler, Richard Portes, Helene Rey, and Pedro Teles for helpful comments and suggestions. All errors remain ours. The views expressed in this paper are those of the authors and do not necessarily reflect the position of the Deutsche Bundesbank.
default, i.e., it is optimal for the government to occasionally repay less than what is legally required to serve its outstanding debt contracts. This is true even if default events give rise to sizable deadweight costs. A quantitative analysis suggests that optimal default policies significantly increase ex-ante welfare, relative to a situation where sovereign default is simply ruled out by assumption.

The fact that sizable welfare gains can arise from sovereign default may appear surprising, given that policy discussions and also the academic literature tend to emphasize the inefficiencies associated with sovereign default events. Popular discussions, for example, tend to focus on the potential ex-post costs associated with a sovereign default, say, the adverse consequences for the functioning of the banking sector or the economy as a whole. While certainly relevant, we show that sovereign default can remain optimal, even if default costs of an empirically plausible magnitude arise. Likewise, following the seminal contribution of Eaton and Gersovitz (1981), much of the academic literature tends to emphasize the inefficiencies created by default decisions: anticipation of default in the future limits the ability to issue debt today, thereby constrains the ability to smooth out adverse shocks.

Our analysis emphasizes that sovereign default fulfills also a very useful economic function, even in a setting with a fully committed government: a default engineers a resource transfer from lenders to the sovereign debtor in times when resources are scarce on the sovereign’s side. The option to default thus provides insurance against adverse economic developments in domestic income. This point has previously been emphasized by Grossmann and Van Huyck (1988), who coined the term ‘excusable default’ to capture default events that are the result of an implicit risk sharing agreement between the sovereign borrower and its lenders. Assuming the absence of default costs, Grossman and Van Huyck (1988) study whether the optimal allocation with ‘excusable’ and cost-free default can be sustained as a reputational equilibrium in a setting without a committed lender. The present analysis abstracts entirely from issues related to lack commitment, instead is concerned with characterizing the optimal allocation with ‘excusable’ default, but for the empirically more plausible setting with non-zero default costs. As we prove analytically, the presence of default costs strongly affects optimal allocations and the optimal default policy, with default policies being discontinuously affected when moving zero to positive default cost levels.

Sovereign default is Ramsey optimal in our setting because government bond markets are incomplete, so that international bond markets do not provide any explicit insurance against domestic income shocks. The incompleteness of government bond markets thereby emerges endogenously from the presence of contracting frictions, that we describe in detail in section 3 of the paper. These frictions make it optimal for the government to issue debt contracts that - in legal terms - promise a repayment amount that is not contingent on future events. This is in line with empirical evidence, which shows that existing government debt consist predominantly of non-contingent debt instruments. The contracting framework represents an important

1 Moreover, adjustments of the domestic investment margin only partially contribute to smoothing domestic consumption.

2 Most sovereign debt is non-contingent in nominal terms only, and could be made contingent by adjusting the price level, a point emphasized by Chari, Kehoe and Christiano (1991). As shown
advance over earlier work studying Ramsey optimal government policy under commitment and incomplete markets, which simply assumes that government bond markets are incomplete (e.g., Sims (2001), Angeletos (2002), Ayiagari et al. (2002), or Adam (2011)). The contracting framework also provides microfoundations for the presence of ‘default costs’.

Using this setting with non-contingent sovereign debt and default costs, we extend the existing Ramsey policy literature by treating repayment of debt as a (continuous) decision variable in the optimal policy problem. We show analytically that for a wide range of default cost specifications the assumption of full debt repayment is inconsistent with fully optimal behavior. While full repayment is optimal if the country has accumulated a sufficient amount of international wealth, which then serves as a buffer against adverse domestic shocks, full repayment is suboptimal for sufficiently low wealth levels and for at least one productivity realization, provided default costs do not take on prohibitive values.\(^3\) The presence of non-zero default costs is key for this finding, as the optimal default patterns would otherwise be entirely independent of the country’s wealth position.

Besides providing analytical characterizations of the optimal default policies, we also seek to quantitatively address under what economic conditions sovereign default is part of Ramsey optimal policy. For this purpose, we provide a lower bound estimate for the costs of default implied by our structural model and use it as an input for our quantitative analysis. We show that plausible levels of default costs make it optimal for the government not to default following business cycle sized shocks to productivity, thereby vindicating the full repayment assumption often entertained in the Ramsey policy literature with incomplete markets. Only when the country’s net foreign debt position approaches its maximum sustainable level, does sovereign default become optimal following an adverse business cycle shock.

Given that reasonably sized default costs largely eliminate sovereign default in response to business cycle sized shocks, we introduce economic ‘disaster’ risk into the aggregate productivity process, following Barro and Jin (2011). Default then reemerges as part of optimal government policy, following the occurrence of a disaster shock. This is the case even for sizable default costs and even when the country’s net foreign asset position is far from its maximally sustainable level. It continues to be optimal, however, not to default following business cycle sized shocks to aggregate productivity, as long as the country’s international wealth position is not too close to its maximal sustainable level.

We also investigate the welfare consequences of using government default by comparing the optimal policy with default to a situation where the government is assumed to repay debt unconditionally. In the latter setting, adjustments in the international wealth position and of the domestic investment margin are the only channels avail-

\(^3\) Default costs are prohibitive if the costs of default are equal to, or higher than the amount of resources that is not repaid to lenders.
able for smoothing domestic consumption. The consumption equivalent welfare gains associated with optimal default decisions easily reach one percentage point of consumption each period, even when sizable costs associated with a government debt default.

In related work, Sims (2001) discusses fiscal insurance in the context of whether or not Mexico should dollarize its economy. Considering a setting where the government is assumed to issue only non-contingent nominal debt that is assumed to be repaid always, he shows how giving up the domestic currency allows for less insurance, as it deprives the government of the possibility to use price adjustments to alter the real value of outstanding debt. The present paper considers a model with real bonds that are optimally non-contingent and allows for outright government debt default. Our setting could thus be reinterpreted as one where bonds are effectively non-contingent in nominal terms, but where the country has delegated the control of the price level to a monetary authority that pursues price stability, say by dollarizing or by joining a monetary union. As we then show, in such a setting the default option still provides the country with a possible and quantitatively relevant insurance mechanism.

Angeletos (2002) explores fiscal insurance in a closed economy setting with exogenously incomplete government bond markets, assuming also full repayment of debt. He shows how a government can use the maturity structure of domestic government bonds to insure against domestic shocks, by exploiting the fact that bond yields of different maturities react differently to shocks. This channel is unavailable in our small open economy setting, since the international yield curve does not react to domestic events.

The remainder of the paper is structured as follows. Section 2 introduces the economic environment, formulates the Ramsey policy problem, and derives the necessary and sufficient conditions characterizing optimal policy. To simplify the exposition, this section assumes that the government issues non-contingent debt only and that deviations from the legally stated repayment promise gives rise to proportional default costs. Section 3 then endogenizes the government debt contract and derives the optimality of non-contingent government debt and the presence of default costs from a specific contracting model. Section 4 presents a number of analytical results characterizing optimal default policies. In section 5 we quantitatively evaluate the model predictions by studying optimal default policies in a setting with business cycle sized shocks. Section 6 then introduces economic disaster shocks and discusses their quantitative implications. Section 7 studies the welfare implications of using the default option and section 8 discusses an extension of the model to bonds with longer maturity. A conclusion briefly summarizes. Technical material is contained in a series of appendices.

2 A Small Open Economy Model

Consider a small open economy with shocks to domestic productivity where the government can internationally borrow and invest to insure domestic consumption against fluctuations in domestic income. The economy is populated by a representa-
tive consumer with expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

(1)

where $c \geq 0$ denotes consumption and $\beta \in (0, 1)$ the discount factor. We assume $u' > 0$, $u'' < 0$, and that Inada conditions hold. Domestic output is produced by a representative firm using the production function

$$y_t = z_t k_{t-1}^\alpha - \bar{c},$$

where $y_t$ denotes output of consumption goods in period $t$, $\bar{c} \geq 0$ some fixed expenditures, $k_{t-1}$ the capital stock from the previous period, $\alpha \in (0, 1)$ the capital share, and $z_t > 0$ an exogenous stochastic productivity shock. Productivity shocks are the only source of randomness in the model and cause domestic income to be risky. Productivity assumes values from some finite set $Z = \{z^1, ..., z^N\}$ with $N \in \mathbb{N}$ and the transition probabilities across periods are described by some measure $\pi(z'|z)$ for all $z', z \in Z$. Without loss of generality, we order productivity states such that $z^1 > z^2 > ... > z^N$. The fixed expenditures $\bar{c} \geq 0$ can either be interpreted as an output component that is consumed as a fixed cost in the production process, or - as we prefer - as a fixed subsistence level for consumption expenditures. In the latter case, $y_t$ denotes output in excess of this subsistence level.\(^4\) What is important is that $\bar{c}$ is an output component that cannot be transferred to international lenders. In our quantitative analysis we calibrate $\bar{c} \geq 0$ in a way to obtain reasonably tight international borrowing limits for the domestic economy.

### 2.1 The Government

The government seeks to maximize the utility of the representative domestic household (1) and is fully committed to its plans. It can insure consumption against domestic income risk by investing in foreign bonds, i.e., by building up a buffer stock of foreign wealth, and by issuing own bonds, i.e., by borrowing internationally.\(^5\)

Without loss of generality, we consider a setting in which foreign bonds are zero coupon bonds with a maturity of one period.\(^6\) Foreign bonds are assumed to be risk free and the interest rate $r$ on these bonds satisfies $1 + r = 1/\beta$. We let $F_t \geq 0$ denote the government’s holdings of foreign bonds in period $t$. These bonds mature in period $t + 1$ and repay $F_t$ units of consumption at maturity.

We furthermore assume that the domestic government has a specific ‘technology’ available for issuing domestic bonds, i.e., for borrowing internationally. We provide

---

\(^4\)This is consistent with the utility specification in equation (1) if we set $u(c) = -\infty$ for all $c < 0$, i.e., whenever consumption falls short of its subsistence level.

\(^5\)For the contracting model presented in section 3, it is actually optimal that the government borrows internationally on behalf of private agents. Alternatively, one may assume that private agents do not have access to the international capital markets.

\(^6\)Allowing for a richer maturity structure for foreign bonds makes no difference for the analysis: the small open economy setting implies that foreign interest rates are independent of domestic conditions, so that the government cannot use the maturity structure of foreign bonds to insure against domestic productivity shocks.
microfoundations for our ‘technology’ assumptions in section 3 below using an explicit contracting framework.

We assume that the government can issue non-contingent one period zero coupon bonds only. This means that domestic bonds promise - as part of their legally stated payment obligation - to unconditionally repay one unit of consumption one period after they have been issued. The effects of introducing domestic bonds with longer maturity will be discussed separately in section 8. The government can choose to deviate from this legally stated payment obligation, but such deviations are costly. Specifically, the government can determine - at the time the bonds are issued - in which future states of nature repayment will fall short of the legally stated amount and by how much. The government thus chooses in which states there will be a sovereign default, as well as the size of the default. Default events, however, give rise to ‘default costs’, which take the form of a dead-weight resource cost. This captures the intuitive fact that sizable ex-post costs can be associated with a sovereign default event.

Let $D_t \geq 0$ denote the amount of domestic bonds issued by the government in period $t$. These bonds legally promise to repay $D_t$ units of consumption in period $t+1$. When issuing these bond in period $t$, the government also decides on a default profile $\Delta_t \in [0,1]^N$, which is a vector determining for each future productivity state $z_n (n = 1, ..., N)$ what share of the legal payment promise the government will default on

$$\Delta_t = (\delta_t^1, ..., \delta_t^N).$$

An entry of one indicates a state in which full default occurs, an entry of zero a state with full repayment, and intermediate values capture partial default events. Let $\delta_t(z_{t+1})$ denote the entry in the default profile $\Delta_t$ pertaining to productivity state $z_{t+1} \in Z$. Total repayment on domestic bonds maturing in period $t + 1$ is then given by

$$D_t(1 - \delta_t(z_{t+1})) + \lambda D_t \delta_t(z_{t+1}).$$

(2)

The first term captures the amount of domestic debt that is repaid to lenders, net of the default share $\delta_t(z_{t+1})$; the second terms captures the default costs accruing to the sovereign borrower, where $\lambda \geq 0$ is a cost parameter. Default cost only emerge if $\delta_t(z_{t+1}) > 0$ and are assumed to be proportional to the default amount $D_t \delta_t(z_{t+1})$. We consider proportional default costs mainly for analytical convenience, as such a specification allow us to prove concavity of the Ramsey problem later on. While it may be plausible that sovereign default events also give rise to fixed costs that are independent of the default amount, such specifications generate non-convexities in the constraint set of the Ramsey problem, which considerably complicate the optimal policy analysis. Our proportional specification is furthermore similar to the specifications used in Zame (1993) and Dubey, Geanakoplos and Shubik (2005) who previously introduced proportional default costs to study default on private contracts.\(^7\)

\(^7\)Default costs in our setting represent a resource cost, while the general equilibrium literature with incomplete markets referenced above introduces default cost in the form of a direct utility cost, which enters separably into the borrower’s utility function. The resource cost specification is more natural given the microfoundations we provide in section 3, but we conjecture that imposing a direct
In the setting just described, the government can insure domestic consumption against productivity risk either by adjusting its holdings of foreign and domestic bonds, i.e., by adjusting its buffer stock of savings or debt, by choosing appropriate default policies on domestic bonds, or by adjusting domestic investment. The optimal mix between these insurance mechanisms will depend on the level of the default costs $\lambda$.

### 2.2 The Ramsey Problem

To derive the Ramsey problem determining optimal government policies, it turns out to be useful to define the amount of resources available to the domestic government at the beginning of the period, i.e., before issuing new domestic debt, before making investment decisions and before paying for fixed expenditures, but after (partial) repayment of maturing bonds.\(^8\) We refer to these resources as beginning-of-period wealth and define them as

$$w_t = z_t k_{t-1}^{\alpha} + F_{t-1} - D_{t-1}(1 - (1 - \lambda)\delta_{t-1}(z_t)).$$

Beginning-of-period wealth is a function of past decisions and of current exogenous shocks only. The government can raise additional resources in period $t$ by issuing new domestic bonds, and use the available funds to invest in foreign riskless bonds, in the domestic capital stock, to finance consumption, and to pay for the fixed expenditures $\bar{c}$. The economy's budget constraint is thus given by

$$w_t + \frac{D_t}{1 + R(z_t, \Delta_t)} = c_t + \bar{c} + k_t + \frac{F_t}{1 + r},$$

where $1/(1 + r)$ and $1/(1 + R(z_t, \Delta_t))$ denote the issue price of the foreign and domestic bond, respectively. The domestic interest rate $R(z_t, \Delta_t)$ thereby depends on the default profile $\Delta_t$ chosen by the government and on the current productivity state, as the latter generally affects the likelihood of entering different states tomorrow. Due to the small open economy assumption, the government can take the pricing function $R(\cdot, \cdot)$ as given in its optimization problem. Assuming that international investors are risk-neutral, this pricing function is given by

$$\frac{1}{1 + R(z_t, \Delta_t)} = \frac{1}{1 + r} \sum_{n=1}^{N} (1 - \delta_t(z^n)) \cdot \pi(z^n | z_t),$$

which equates the expected returns on the domestic bond and the foreign bond.

Using the previous notation, the Ramsey problem characterizing optimal govern-

---

\(^8\)Below we do not distinguish between the government budget and the household budget, instead consider the economy wide resources that are available. This implicitly assumes that the government can costlessly transfer resources between these two budgets, e.g., via lump sum taxes.
ment policy is then given by

\[
\max \quad \{ F_t \geq 0, D_t \geq 0, \Delta_t \in [0,1], k_t \geq 0, \alpha_t \geq 0 \} \quad E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (5a)
\]

s.t.:

\[
c_t = w_t - c_t - k_t + \frac{D_t}{1 + \bar{r}(z_t, \Delta_t)} - \frac{F_t}{1 + r} \quad (5b)
\]

\[
w_{t+1} \geq NBL(z_{t+1}) \quad \forall z_{t+1} \in Z \quad (5c)
\]

\[
w_0, z_0 : \text{given.}
\]

We impose the natural borrowing limits (5c) on the problem to prevent the possibility of explosive debt dynamics (Ponzi schemes). We allow the natural borrowing limits to be potentially state contingent and assume that the initial condition satisfies \( w_0 \geq NBL(z_0) \). Note that the time-zero optimal Ramsey policy involves defaulting on all outstanding debt at time zero, a feature that should be reflected in the initial value for \( w_0 \).

While intuitive, the Ramsey problem (5) is characterized by two features that complicate its solution. First, the price of the domestic government bond in the constraint (5b) depends on the chosen default profile, so that the constraint fails to be linear in the government’s choice variables. It is thus unclear whether problem (5) is concave, which prevents us from working with first order conditions. Second, the presence of the natural borrowing limits (5c) creates problems for numerical solution algorithms. Specifically, imposing sufficiently lax natural borrowing limits, as is usually recommended if one wants to rule out Ponzi schemes only, gives rise to a non-existence problem: sufficiently lax borrowing limits imply that there exist beginning-of-period wealth levels above these limits, for which no policy can insure that the borrowing limits are respected under all contingencies. This creates problems for numerical solution approaches and thus for a quantitative evaluation of the model. While one could remedy the existence problem by imposing sufficiently tight borrowing limits, such an approach could imply that one rules out feasible and potentially optimal policies that would be consistent with non-explosive debt dynamics.

In the next sections address both of these issues in turn. We first prove concavity of the Ramsey problem by reformulating it into a specific variant of a complete markets model, which can be shown to be concave and equivalent to the original problem. This approach to proving concavity is - to the best of our knowledge - new to the literature and should be useful in a range of other applications involving default decisions. We then proceed by showing how to properly deal with the presence of natural borrowing limits in numerical applications. Again, this approach seems new to the literature and of interest for a range of other applications. In a final step, we show that the concave and equivalent formulation of the Ramsey problem has a recursive structure, which greatly facilitates numerical solution.

### 2.2.1 Concavity of the Ramsey Problem

We now define an alternative Ramsey problem with a different asset market structure. As we show, this alternative problem is equivalent to the original problem (5). Since
the alternative problem is concave, we can work with first order conditions.

Consider a setting in which the government can trade each period $N$ Arrow securities and a single riskless bond. All assets have a maturity of one period. The vector of Arrow security holdings in period $t$ is denoted by $a_t \in \mathbb{R}^N$ and the $n$-th Arrow security pays one unit of output in $t+1$ if productivity state $z^n$ materializes. The associated price vector is denoted by $p_t \in \mathbb{R}^N$. Given the risk-neutrality of international lenders, the price of the $n$-th Arrow security in period $t$ is

$$p_t(z^n) = \frac{1}{1 + r} \pi(z^n | z_t). \quad (6)$$

Let $b_t$ denote the country’s holdings of riskless bonds in period $t$. As before, the interest rate on riskless instruments is $1+r$ and these bonds mature in $t+1$. Beginning-of-period wealth can then be expressed as

$$\tilde{w}_t \equiv z_t \bar{k}_{t-1}^\alpha + b_{t-1} + (1 - \lambda)a_{t-1}(z_t), \quad (7)$$

where $a_{t-1}(z_t)$ denotes the amount of Arrow securities purchased for state $z_t$, $\bar{k}_{t-1}$ capital invested in the previous period, $b_{t-1}$ the bond holdings from the previous period, and $\lambda \geq 0$ the parameter capturing potential default costs in the original problem (5). Note that the Arrow securities in equation (7) pay out only $1 - \lambda$ units of consumption to the holder of the asset, but are priced by the issuer in equation (6) as if they would pay one unit of consumption. This wedge will capture the presence of default costs. Since Arrow securities can be used to replicate the payout of the riskless bond, the price system - as perceived by the domestic sovereign - is not arbitrage free whenever $\lambda > 0$. To have a well-defined problem, we therefore impose the additional constraint $a \geq 0$, while leaving $b$ unconstrained. Intuitively, the restriction $a \geq 0$ insures that the country cannot ‘create’ additional resources in the form of negative default costs by going short in the Arrow securities.

The Ramsey problem for this alternative asset structure is then given by:

$$\begin{align*}
\max_{\{b_t, a_t \geq 0, \bar{k}_t \geq 0, \bar{c}_t \geq 0\}} & \quad E_0 \sum_{t=0}^{\infty} \beta^t u(\bar{c}_t) \\
\text{s.t.} & \quad \bar{c}_t = \tilde{w}_t - \bar{c} - \bar{k}_t - \frac{1}{1 + r} b_t - p_t a_t \\
& \quad \tilde{w}_{t+1} \geq NBL(z_{t+1}) \quad \forall z_{t+1} \in \mathcal{Z} \\
& \quad \tilde{w}_0 = w_0, \ z_0 \text{ given.} 
\end{align*} \quad (8a)$$

Problem (8) has the same concave objective function as problem (5) and imposes the same natural borrowing limits. Importantly, however, the constraint (8b) is now linear in the choice variables, so that first order conditions (FOCs) provide necessary and sufficient conditions for optimality.\footnote{This follows from the additional observation that future beginning of period wealth, as defined in equation (7), is a linear function of the financial market choices $(a, b)$ and a convex function of investment $k$.} The necessary and sufficient FOCs of problem (8)
can be found in appendix A.1. Appendix A.2 then proves the following equivalence result:

**Proposition 1** A consumption path \( \{c_t\}_{t=0}^{\infty} \) is feasible in problem (5) if and only if the consumption path \( \{\tilde{c}_t\}_{t=0}^{\infty} \), with \( \tilde{c}_t = c_t \) for all \( t \geq 0 \), is feasible in problem (8).

The proof of proposition 1 shows how the financial market choices \( \{b_t, a_t\} \) supporting a consumption allocation in problem (8) can be translated into financial market and default choices \( \{F_t, D_t, \Delta_t\} \) supporting the same consumption allocation in the original problem (5), and vice versa. The relationship between these set of choices is given by

\[
\begin{align*}
 b_t &= F_t - D_t \\
 a_t &= D_t \Delta_t.
\end{align*}
\]

The riskless bond position \( b \) in problem (8) can thus be interpreted as the net foreign asset position in problem (5), while the Arrow security holdings \( a \) in problem (8) can be interpreted as the state contingent default decisions on domestic bonds. We will make use of this interpretation in the latter part of the paper, as we solve the simpler problem (8), but interpret the solution in terms of the financial market choices for the original problem (5) with default. Moreover, to support the same consumption allocation in problems (8) and (5) requires identical investment decisions, i.e., \( \tilde{k}_t = k_t \) for all \( t \geq 0 \), which allows us to use these variables interchangeably.

### 2.2.2 Dealing with Natural Borrowing Limits

In our quantitative evaluation of the model, we wish to impose borrowing limits that insure existence of optimal policies, but that are sufficiently lax to not rule out policies that would be consistent with non-explosive debt dynamics. We call such borrowing limits the ‘marginally binding natural borrowing limits’. We explain below how one can compute them and derive their properties.

Let \( NBL(z^n) \) denote the marginally binding natural borrowing limit (NBL) in productivity state \( z^n, n = 1, \ldots, N \). It is defined by the following optimization problem\(^{10}\)

\[
NBL(z^n) = \arg \min w^0(z^n) \ s.t. \ \\ w^0(z^j) \geq NBL(z^j) \text{ for } j = 1, \ldots, N,
\]

where \( w^0(z^n) \) denotes beginning-of-period wealth in state \( z^n \) and \( w^0(z^j) \) the beginning-of-period wealth in the next period if next period’s productivity is \( z^j \). Marginally binding NBLs can thus be interpreted as a set of state-contingent minimum beginning-of-period wealth levels, such that beginning-of-period wealth in all future states remains above these same limits.\(^{11}\) From problem (11) it becomes clear that the marginally binding NBLs are implicitly defined by a fixed point problem.

\(^{10}\)A more explicit formulation of the problem is provided in (35) in appendix A.3.\(^{11}\)These limits depend only on the current productivity shock because the shock process is Markov and because beginning-of-period wealth is the only other state variable, as will become clear in section 2.2.3.
The fixed point problem (11) is non-trivial because the optimization problem it contains admits for a considerable number of corner solutions, due to the presence of linear components in the constraints and objective function, and due to presence of inequalities constraints for the choice variables.\(^{12}\) In numerical solution approaches, it is in principle possible to check all possible corners, each of which gives rise to a set of possible borrowing limits \(NBL(z^n)\) \((n = 1, ..., N)\) solving the fixed point problem implicitly defined by (11). Although we never encountered such a situation in our numerical applications, it unclear whether there exists one corner solution that provides the \textit{uniformly} lowest borrowing limit for all productivity states \(z^n\. In general, one corner may imply a tighter borrowing limit for one productivity state than another corner, but the latter may imply a laxer limit for another productivity state. In such a situation it would be unclear which set of marginally binding NBLs one should impose. To overcome this potential problem, it is helpful to impose the following mild regularity condition:\(^{13}\)

\textbf{Condition 1} \textit{The productivity process }\(\pi(\cdot | \cdot)\textit{ is such that lower productivity states are associated with tighter borrowing limits:}

\[ NBL(z^1) \leq NBL(z^2) \leq ... \leq NBL(z^N) \]  

As we show below, regularity condition (12) insures that there exists a unique set of possible borrowing limits \(NBL(z^n)\) \((n = 1, ..., N)\) solving the fixed point problem implicitly defined (11). Regularity condition (12) is satisfied, for example, when productivity states are iid or for the polar case where productivity states display sufficiently high persistence.\(^{14}\) For all of our calibrated productivity processes, we find that the regularity condition (12) holds.\(^{15}\) Overall, the regularity condition (12) is of interest because of the following important result:

\textbf{Proposition 2} \textit{If (12) holds, then there exists, generically for all model parameterizations, a unique solution to the fixed point problem implicitly defined by (11).}

The proof of the proposition can be found in appendix A.3. The proof is constructive, i.e., it also explains how the NBLs can actually be computed. The following result then shows that the unique fixed point solution to (11) indeed defines the loosest borrowing limits consistent with non-explosive debt dynamics:

\(^{12}\)This can be seen from the more explicit formulation of (11) provided in equation (35) in appendix A.3.\(^{13}\)The regularity condition is somewhat tighter than what is actually needed, but the less tight formulation is notationally more burdensome, prompting us to stick to the simpler formulation presented above.\(^{14}\)In the former case, the optimization problems (11) are identical for all states \(z^n\) \((n = 1, ..., N)\), so that (12) must hold with strict equality at the fixed point. In the latter case, (12) holds with strict inequality when states are perfectly persistent, due to assumed ordering \(z^1 > z^2 > ... > z^N\). This continues to be true if the likelihood of transiting into other states is sufficiently small, as buying insurance for such states to satisfy the borrowing limits is then extremely cheap, see equation (6), and will not lead to a reordering of the borrowing limits.\(^{15}\)In our numerical applications we also check for possible alternative solutions to (11) that would not satisfy (12).
Proposition 3 Suppose condition (12) holds. Given a productivity state $z^n$, with $n \in \{1, \ldots, N\}$, and a beginning-of-period wealth level $\tilde{w}$:

1. If $\tilde{w} \geq NBL(z^n)$, then there exists a policy that is consistent with non-explosive debt dynamics along all future contingencies.

2. If $\tilde{w} < NBL(z^n)$ then there exists no policy that does not violate any finite debt limit with positive probability.

The proof of proposition 3 is in appendix A.4.

2.2.3 Recursive Formulation of the Ramsey Problem

We now show that the Ramsey problem (8) has a recursive structure. This is of interest because it allows expressing - without loss of generality - the optimal policies as functions of a small number of state variables. Let $V(\tilde{w}_t, z_t)$ denote the value function associated with optimal continuation policies when starting with beginning of period wealth $\tilde{w}_t$ and productivity state $z_t$. The Ramsey problem (8) then has a recursive representation given by:

$$V(\tilde{w}_t, z_t) = \max_{b_t, a_t \geq 0, k_t \geq 0} u(\tilde{w}_t - c - \tilde{k}_t - \frac{1}{1 + r}b_t - p_t(a_t) + \beta E_t[V(\tilde{w}_{t+1}, z_{t+1})]
\text{s.t. } \tilde{w}_{t+1} = z_{t+1}k_{t+1}^0 + b_t + (1 - \lambda)a_t(z_{t+1}) \\
\tilde{w}_{t+1} \geq NBL(z_{t+1}) \forall z_{t+1} \in Z.$$ 

We can thus express policies as functions of the two state variables $(\tilde{w}_t, z_t)$.

3 Endogenously Incomplete Government Debt Markets

This section provides explicit microfoundations for the previously made assumptions that the government issues non-contingent debt only and that deviations from the legally stated repayment promise gives rise to default costs. We do so by considering a setting where the government can issue arbitrary state contingent debt contracts, but where contracting frictions make it optimal for the government to issue debt with a non-contingent legal repayment promise only. The same frictions also give rise to default costs. The microfoundations we provide below provide a specific example justifying the setup specified in the previous section, but a range of other conceivable microfoundations may exist.

Explicit and Implicit Contract Components. We consider a setting where a government debt contract consists of two contract components. The first component is the explicit contract, which is written down in the form of a legal text. In its most general form, the legal text consists of a description of the contingencies $z^n$ and of the legal repayment obligations $l^n \geq 0$ associated with each contingency $n \in \{1, \ldots, N\}$.

$$\text{We normalize the size of the legal contract by assuming max}_n l^n = 1.$$ 

The fact that $l^n \geq 0$ can be justified by assuming lack of commitment on the lenders’ side. Such lack of commitment appears reasonable, given the existence of secondary markets on which government debt can be traded.
The second component is an implicit contract component. This component is not formalized in explicit terms but is commonly understood by the contracting parties. We capture such implicit contract components by a state contingent ‘default profile’ \( \Delta = (\delta^1, \ldots, \delta^N) \in [0,1]^N \), which specifies for each possible contingency the share of the legal payment obligation that is not fulfilled by the government. Actual repayment at maturity is then jointly determined by the explicit and implicit contract components and given by

\[ l^n(1 - \delta^n) \]

for each contingency \( n \in \{1, \ldots, N\} \). If a contingency arises for which \( \delta^n > 0 \), the country pays back less than the legally or explicitly specified amount \( l^n \) and we shall say that ‘the country is in default’. The explicit and implicit contract components are perfectly known to agents.

In the setting just described, a desired state-contingent repayment profile can be implemented by incorporating it either into the explicit legal repayment profile \( l^n \) or into the implicit profile \( \delta^n \). Absent further frictions, these two components would be perfect substitutes and the optimal form of the government debt contract thus indeterminate.

**Contracting Frictions.** We now introduce two simple contracting frictions. First, we assume that explicit legal contracting is costly. Second, we assume that implicit contracting, while not creating costs, gives rise to the risk that the common understanding about the implicit contract component may be lost after the maturity date of the contract. The idea underlying this specification is that writing down an explicit legal text requires the input of lawyers, thus consumes resources and is costly, but also insures that there exists a common understanding between the contracting parties independently of time: agents can always go back and read about their contract obligations. This is different for implicit contract components, which agents may have difficulties recalling or agreeing on, especially after the maturity date of the contract. The fact that the common understanding about the implicit contract components may disappear is thereby perfectly and rationally anticipated by all agents.

We now describe these two frictions in greater detail. We normalize the costs of writing an non-contingent legal contract \( (l^n = 1 \text{ for } n = 1, \ldots, N) \) to zero and assume that incorporating a contingency gives rise to a proportional legal fee \( \lambda \geq 0 \) that is charged against the value of the contingent agreement. This is in line with the casual empirical observation that lawyers typically charge fees that are proportional to the value of the agreements they formulate. In particular, legally incorporating a payment \( l^n \leq 1 \) for some contingency \( z^n \) in the explicit contract, involves the costs

\[ \lambda(1 - l^n) \]

per contract issued, where \( 1 - l^n \) denotes the value of the deviation from the baseline payment of 1.

---

17 The fact that \( \delta^n \leq 1 \) can again be justified by lack of commitment on the lenders’ side, which makes it impossible to write contracts that specify additional transfers to the borrower at maturity. The assumption that \( \delta^n \geq 0 \) facilitates interpretation in terms of default, but is never binding in our numerical applications.
While incorporating a state contingency in the repayment structure via the implicit contract component $\delta^n > 0$ does not give rise to legal costs, it exposes the government to the risk that the common understanding about a default event may be lost after the maturity date of the contract. This is relevant for the borrower because in the absence of a recallable implicit contract component, courts base their decisions on a comparison of the explicit contract obligation with the actual actions (payments) that occurred. Default events that are followed by a lack of common understanding about the implicit contract thus provide strong incentives for lenders to sue the government for fulfillment of the explicit contract, i.e., to sue the government for repayment of the legally stated amount.\(^{18}\) Anticipating such behavior, the government will engage - at the time the default occurs - in a negotiation process with the lender, with the objective to reach an explicit legal settlement that protects it from being sued in the future.

The settlement agreement transforms the thus far only implicitly existing contract component into an explicit one by stating that the debt contract is regarded as fulfilled, even if the actual payment amount fell short of the amount specified in the legal text of the contract. The threat of going to court to obtain such an explicit settlement via a court ruling in the period where the default happens and where a common understanding about the implicit components still exists, will induce the lender to agree to such an agreement.\(^{19}\) Since we assume explicit legal contracting to be costly, the settlement agreement following a default event gives rise to the legal costs (or default costs)

$$\lambda^n \delta^n$$

per contract, where $l^n \delta^n$ denotes the value of the settlement agreement, i.e., the defaulted amount on each contract. For simplicity, we assume here that the same proportional fee $\lambda$ that applies to writing an explicit contingent contract ex-ante also applies to the ex-post settlement stage. We discuss below the case where the ex-post settlement costs are higher. While the legal fees associated with writing a legal contract are assumed to be born by the government, we allow for the possibility that the settlement fees are shared between the lender and the borrower, with the lender paying $\lambda^l \geq 0$, the borrower paying $\lambda^b \geq 0$, and $\lambda^l + \lambda^b = \lambda$.

**Optimal Government Debt Contract.** Consider a government that wishes to implement a contingent payment $p(z) \leq 1$ for some contingency $z \in Z$. Specifying

\(^{18}\)As documented in Panizza, Sturzenegger and Zettelmeyer (2009), legal changes in a range of countries in the late 1970's and early 1980's eliminated the legal principle of 'sovereign immunity' when it comes to sovereign borrowing. Specifically, in the U.S. and the U.K. private parties can sue foreign governments in courts, if the complaint relates to a commercial activity, amongst which courts regularly count the issuance of sovereign bonds. We implicitly assume that lenders cannot commit to not sue the government. Again, this appears plausible, given that secondary markets allows initial buyers of government debt to sell the debt instruments to other agents.

\(^{19}\)The fact that - due to the large number of actors involved - the implicit contract component of government debt can be verified in court makes government debt contracts special. Implicit components of private contracts, for example, are often private information available to the contracting parties only, thus cannot be verified in court, not even over the lifetime of the contract. The optimal form of private contracts will therefore generally differ from the optimal form of government debt contracts.
the contingency as part of the legal contract involves the contract writing costs

\[ \lambda (1 - p(z)) \]

per contract and no ex-post settlement costs in case the contingency arises in the future. Alternatively, not specifying the contingent payment as part of the legal contract, gives rise to expected default costs \( \Pr(z|z_0) \lambda (1 - p(z)) \),

where \( z_0 \) is the contingency prevailing at the time when the contract is issued. Since \( \Pr(z|z_0) \leq 1 \) and since default costs are born at a later stage, i.e., when the contract matures, the government will always strictly prefer to issue a non-contingent explicit contract and to shift contingencies into the implicit contract profile. This continues to be true even in the more general case where the ex-post settlement costs are much higher than the cost associated with incorporating the contingency ex-ante into the legal contract, provided the probability \( \Pr(z|z_0) \) of reaching the default event is sufficiently small.

Summing up, it is optimal for the government to issue debt that is non-contingent in explicit legal terms. At the same time, the government has the option to deviate from the legally specified payment amount, but such actions give rise to proportional default costs \( \lambda \). The contracting frictions introduced above thus microfound the assumptions entertained in the previous section.

**Government versus Private Debt.** The contracting framework introduced above can also be used to justify why the government optimally borrows on behalf of private agents in the international market. This is relevant because it allows us to genuinely speak of a sovereign default, i.e., one cannot interchangeably speak of a private default. The optimality of sovereign borrowing emerges because there exists a fundamental difference between sovereign and private debt contracts: the implicit contract components of private contracts are private information to the contracting parties, thus cannot be verified in court, not even over the lifetime of the contract. This is different for a sovereign debt contract which is widely shared between many individuals. Achieving state contingency in private contracts thus has to rely on explicit contracting, which is costly, as implicit private contracting is not self-enforcing.\(^{21}\) To economize on the explicit contracting costs in private debt contract makes it optimal to issue sovereign debt contracts with implicit contract components.

## Optimal Sovereign Default: Analytic Results

This section presents a number of analytic results characterizing the optimal default policies that solve the Ramsey problem (8). We first consider - for benchmark purposes - a setting without default costs (\( \lambda = 0 \)). As we show, the full repayment

\(^{20}\)The expected settlement cost for the lender enter the borrower's optimization reasoning because the borrower has to compensate the lender ex-ante for the expected costs born by the lender.

\(^{21}\)Alternatively, implicit private contracting may rely on self-enforcement in a long-term relationship, from which we abstract here.
assumption is then suboptimal under commitment and sovereign default is optimal for virtually all productivity realizations. This holds true independently of the country’s net foreign asset position. Second, we show that for ‘prohibitive’ default cost levels with $\lambda \geq 1$, default is never optimal. Again this holds independently of the country’s international wealth position. Finally and most interestingly, we present analytic results covering cases with intermediate levels of default costs ($0 < \lambda < 1$). Ramsey optimal default decisions then depend on the country’s wealth level and on the productivity realization. As we show, there exists a discontinuity in the optimal default policies as one moves from $\lambda = 0$ to $\lambda > 0$.

4.1 Zero Default Costs

In the absence of default costs ($\lambda = 0$) the original Ramsey problem (5) reduces to a generalized version of the problem analyzed in section II in Grossman and Van Huyck (1988). The proposition below shows that - as in Grossman and Van Huyck - full consumption smoothing is then optimal, so that the optimal consumption allocation is the same as in a complete markets setting. The result below also characterizes the optimal default and investment policies; the proof can be found in appendix A.5.

**Proposition 4** For $\lambda = 0$ the solution to the Ramsey problem (8) involves constant consumption equal to

$$\tilde{c} = (1 - \beta)(\Pi(z_0) + \tilde{w}_0)$$

where $\Pi(\cdot)$ denotes the maximized expected discounted profits from production, defined as

$$\Pi(z_t) \equiv E_t \left[ \sum_{j=0}^{\infty} \beta^j (-k^*(z_{t+j}) + \beta z_{t+j+1}(k^*(z_{t+j}))^\alpha - \tilde{c}) \right]$$

with

$$k^*(z_t) = (\alpha \beta E(z_{t+1}|z_t))^{1-\alpha}$$

denoting the optimal investment policy. For any period $t$, the optimal default level satisfies

$$a_{t-1}(z_t) \propto - (\Pi(z_t) + z_t(k^*(z_{t-1}))^\alpha)$$

Let $N_t \in \{1, \ldots, N\}$ denote the number of productivity states in $t$ that can be reached from $z_{t-1}$ in $t-1$, according to the transition matrix $\Pi(\cdot|\cdot)$. Since $a_t \geq 0$, it follows from equation (15) that the optimal commitment policy generically involves default for at least $N_t - 1$ productivity realizations in $t$. Default thereby insures the domestic economy against two sources of risk: first, it insures against a low realization of current output due to a low value of current productivity, as captured

---

22Grossman and Van Huyck consider an endowment economy with iid income risk, which is a special case of our setting with production and potentially serially correlated productivity shocks.

23Default is not required for states $z_t$ achieving the maximal value for $\Pi(z_t) + z_t(k^*(z_{t-1}))^\alpha$ across all $z_t \in Z$. For such states default can be set equal to zero, with default levels being strictly positive for all other states. This, however, is not the only possible default pattern implementing full consumption stabilization: one could also choose strictly positive default levels for the states $z_t$ achieving the maximal value for $\Pi(z_t) + z_t(k^*(z_{t-1}))^\alpha$.
by the term \( z_t (k^*(z_{t-1}))^\alpha \) in equation (15), a risk that is present in similar form in the endowment setting of Grossman and Van Huyck (1988); second, it additionally insures the domestic economy against (adverse) news regarding the expected profitability of future investments, as captured by the term \( \Pi(z_t) \). As a result of this policy the ‘net worth’ of the economy, defined as the sum of expected future profits \( \Pi(z_t) \) and accumulated net wealth \( \tilde{w}_t \), remains constant over time and equal to its initial value \( \Pi(z_0) + \tilde{w}_0 \). In the absence of default costs, risk sharing thus fully and exclusively occurs via optimal sovereign default, with net worth remaining constant over time, and domestic investment being at its expected profit maximizing level (14).

To interpret the optimal default patterns implied by proposition 4, suppose that expected future profits \( \Pi(z_t) \) are weakly increasing with current productivity \( z_t \). This is the case whenever \( z_t \) is a sufficiently persistent process, but also if \( z_t \) is iid so that expected future profits are independent of current productivity. Equation (15) then implies that optimal default levels are inversely related to the current level of productivity, i.e., the absolute level of non-repaid claims strictly increases with the distance of current productivity from its maximal level. This pattern is optimal independently of the wealth level of the economy, i.e., is optimal even if the economy has a positive net foreign asset position. With a positive net foreign asset position, the sovereign optimally issues domestic bonds and invests the proceeds into foreign bonds, so as to be able to default on the domestic bonds following adverse shocks, see equations (9) and (10).

### 4.2 Prohibitive Default Costs

We now consider the polar case with prohibitive default cost levels \( \lambda \geq 1 \). Default events then induce deadweight resource costs that (weakly) exceed the amount of resources that the borrower does not repay to lenders. Net of default costs, the sovereign thus cannot gain resources by defaulting. For the equivalent Ramsey problem (8) this implies that the payout from Arrow securities is weakly negative, while the price of Arrow securities for states that can be reached with positive probability is strictly positive, see equation (6). This leads to the following result:

**Lemma 1** For \( \lambda \geq 1 \) it is optimal to choose \( a_t = 0 \) for all \( t \).

For \( \lambda \geq 1 \) it is thus optimal to never use default to insure domestic consumption, instead insurance occurs via the accumulation and decumulation of non-contingent and non-defaultable bonds and potentially via adjustments of the investment margin. An interesting trade-off between default and the adjustment of non-defaultable bond positions thus emerges for a plausible range of intermediate cost specifications with \( 0 < \lambda < 1 \). We investigate such specifications in the next section.

---

24 This follows from the proof of proposition 4 in appendix A.5.
25 The Arrow security choices for future states that are reached with zero probability do not affect welfare, which allows us to set them also equal to zero.
26 The latter is discussed in detail in the next section.
4.3 Intermediate Default Cost Levels

Deriving an analytic solution for the Ramsey policy problem (8) for arbitrary intermediate default cost levels $0 < \lambda < 1$ is generally difficult, as the Ramsey problem is a non-linear dynamic stochastic optimization problem with an endogenous state variable and a number of occasionally binding inequality constraints. Nevertheless, it is feasible to derive analytic solutions to the Ramsey problem when beginning-of-period wealth is either very low or very high. More precisely, we consider below first the case where beginning-of-period wealth level is at its lower bound, i.e., at the marginally binding natural borrowing limit defined in section 2.2.2. In a second step, we derive an approximation to the Ramsey policy that applies for a settings in which beginning-of-period wealth is sufficiently high. Our results below show that the country’s wealth position has an important influence on optimal default policy for intermediate levels of default costs.

4.3.1 Initial Wealth at Its Lower Limit

Consider first a situation where beginning-of-period wealth is at its lower bound $(\bar{w}_t = NBL(z_t))$. We can then define a critical productivity state index $n_t^*$:

$$n_t^* = \arg \max_{n \in [1, \ldots, N]} n \quad \text{s.t.} \quad \sum_{i=n}^{N} \pi(z^i|z_t) \geq 1 - \lambda$$

The critical index $n_t^*$ is defined as the highest productivity index such that reaching states $z^n$ tomorrow with $n \geq n_t^*$ still has a likelihood larger than $1 - \lambda$. It turns out that the index $n_t^*$ divides states for which default is Ramsey optimal tomorrow from states for which full repayment is optimal. The critical index also affects the optimal investment level and the optimal amount of bond holdings. The following proposition characterizes the optimal policy in period $t$ as a function of the critical index $n_t^*$:

**Proposition 5** Suppose the regularity condition (12) holds and $\bar{w}_t = NBL(z_t)$. Then the optimal policy solving the Ramsey policy problem (8) in period $t$ is given by

$$\bar{k}_t = \left( \alpha_\beta \left( \sum_{n=1}^{n_t^*} \pi(z^n|z_t)z^n - \lambda \right) + \sum_{n=n_t^*+1}^{N} \pi(z^n|z_t)z^n \right)^{\frac{1}{1-\lambda}}$$

$$b_t = NBL(z_t) - z_t^\alpha \bar{k}_t^\alpha$$

$$\tilde{c}_t = 0$$

$$a_t(z^n) = 0 \text{ for } n \leq n_t^*$$

$$a_t(z^n) = \frac{NBL(z^n) - z^n \bar{k}_t^\alpha - b_t}{1 - \lambda} > 0 \text{ for } n > n_t^*$$

For $\lambda$ sufficiently close to 1 or $\lambda > 1$ we have have $n_t^* = N$, so that all expressions in proposition 5 are well defined.
The proof of the proposition is given in appendix A.6. Note that the proposition determines optimal policy in period $t$, when beginning-of-period wealth is at its lower bound, but does not determine the optimal policy for other periods. This is possible, even though the underlying optimization problem is a dynamic infinite horizon problem, because the choice set at the marginally binding natural borrowing limit reduces to a singleton.

Equations (20) and (21) jointly show that default is suboptimal for all sufficiently good productivity states $z^n$ with $n \leq n^*_t$. For $n > n^*_t$ strictly positive default is optimal and the default amount is strictly increasing in $n$, i.e., there is more default the lower the productivity realization. Consistent with earlier results, it will never be optimal to default as $\lambda \to 1$, as then $n^*_t \to N$. Conversely, it is optimal to default in all but one states as $\lambda \to 0$, as then $n^*_t \to 1$. Moreover, it follows from problem (16) that an increase in default costs (weakly) reduces the set of states for which default is optimal.

Proposition 5 also shows that consumption is at its lower bound once wealth is at the marginally binding borrowing limit. Consumption will stay at its lower bound in the next period, if a sufficiently bad productivity state is reached, i.e., a state $z^n$ with index $n \geq n^*_t$. This is so because the Arrow security and bond purchases insure that tomorrow’s beginning-of-period wealth levels are exactly at their state-contingent marginally binding natural borrowing limit for all productivity states $z^n$ with $n \geq n^*_t$, so that proposition 5 applies again in the next period. Yet, if a productivity state $z^n$ with $n < n^*$ is reached, then beginning-of-period wealth will strictly exceed the natural borrowing limit and consumption will move back to strictly positive values. Obviously, this can only happen if $\lambda$ is sufficiently large, as otherwise $n^*_t = 1$.

Equation (17) shows that the presence of default costs can also distort the optimal investment decision. As long as $n^*_t = 1$, which is the case for sufficiently low levels of the default cost, the investment margin remains undistorted, i.e., identical to the expected profit maximizing investment level $k^*(z_t)$ defined in equation (14). Yet, once $n^*_t > 1$ there is a downward distortion of total investment relative to $k^*(z_t)$. This downward distortion is increasing in $n^*_t$ and thus in the level of default costs $\lambda$. This shows that default costs not only reduce the opportunities for risk sharing, but also adversely affect investment decisions. As we show in our quantitative application, distorted investment decision can be an important source of welfare losses.

### 4.3.2 Large Wealth Levels

This section derives analytical expressions that approximate the Ramsey optimal policies for sufficiently large beginning-of-period wealth levels. The following position summarizes the main result:

**Proposition 6** Suppose $\lambda > 0$ and consumption preferences satisfy $\lim_{c \to \infty} u''(c)/u'(c) = \cdots$
0. Consider a time horizon $T < \infty$ and for $j = 0, ..., T$ the policies

$$
c_{t+j} = (1 - \beta)(\Pi(z_{t+j}) + \tilde{w}_{t+j})
$$

$$
k_{t+j} = k^*(z_{t+j})
$$

$$
b_{t+j} = (1 + r)(\tilde{w}_{t+j} - k^*(z_{t+j})) - (1 - \beta)(\Pi(z_{t+j}) + \tilde{w}_{t+j}) - \tau)
$$

$$
a_{t+j}(z^n) = 0 \text{ for all } n = 1, ..., N
$$

$$
\omega_{t+j}(z^n) = 0 \text{ for all } n = 1, ..., N
$$

where $\Pi(z_{t+j})$ and $k^*(z_{t+j})$ are as defined in proposition 4. For any $\epsilon > 0$ we can find a wealth level $\tilde{w} < \infty$ so that for all initial wealth levels $\tilde{w}_t < \infty$ satisfying $\tilde{w}_t \geq \tilde{w}$, the Euler equation errors $e_{t+j}$ implied by the policies above satisfy $e_{t+j} < \epsilon$ for all periods $j = 0, ..., T - 1$.

The proof of the proposition is contained in appendix A.7. The fact that the Euler equation error vanish for sufficiently large initial wealth levels $\tilde{w}_t$ implies that the policies stated in the proposition approximate the truly optimal policies increasingly well, see Santos (2000) for example. For a sufficiently large wealth level, domestic investment thus remains undistorted, i.e., maximizes expected discounted profits. In addition, it is optimal to always fully repay debt. This is in stark contrast to the case with $\lambda = 0$ where frequent default is optimal, independently of the wealth level, see proposition 4. There thus exists a discontinuity of optimal default policies at $\lambda = 0$. The discontinuity implies that for sufficiently high wealth levels, the presence of even tiny default costs implies a complete shift from frequent default to no default, so that risk sharing occurs optimally via self-insurance only, i.e., via the accumulation and decumulation of international wealth. This result is true as long as the coefficient of absolute risk aversion in consumption decreases towards zero as consumption increases, e.g., for consumption preferences featuring constant relative risk aversion. Intuitively, vanishing absolute risk aversion causes the output risk of given size implied by optimal investment levels to have only negligible influence on consumption utility, whenever consumption (and thus wealth) are sufficiently high. Since default costs are strictly positive, it becomes then suboptimal to use default to insure against these output fluctuations.

5 Quantum Exploration of Optimal Default Policy

This section investigates whether sovereign default can be optimal in a realistically calibrated version of the model with business cycle sized shocks to domestic productivity. The theoretical results established in the previous section show that sovereign default is suboptimal for sufficiently high wealth levels, but likely to be optimal for low wealth levels. This section analyzes optimal default policies for intermediate wealth levels and determines how high the economy’s net foreign asset position plausibly has to be, so as to cause sovereign default to be suboptimal. To answer this quantitative question, we numerically solve for the optimal default policies in a calibrated version of the model. The next section presents the model calibration, including our estimation of default costs. The resulting optimal default policies are discussed thereafter.
5.1 Model Calibration

We interpret a model period as one year. For the productivity process we use a standard parameterization from the business cycles literature and set the annual persistence of technology to \((0.9)^4\) and the annual standard deviation for the innovation to 1%.

\[^{29}\text{Using Tauchen’s (1986) procedure to discretize into a process with two states, one obtains a high productivity state } z^h = 1.0133, \text{ a low productivity state } z^l = 0.9868 \text{ and a transition matrix}
\]

\[\pi (\cdot | \cdot) = \begin{pmatrix} 0.8077 & 0.1923 \\ 0.1923 & 0.8077 \end{pmatrix}. \tag{22}\]

The capital share parameter in the production function is set to \(\alpha = 0.34\) and the annual discount factor to \(\beta = 0.97\). The latter implies an annual real interest rate of approximately 3% for risk free debt instruments. We choose consumption preferences with constant relative risk aversion

\[u(c) = \frac{c^{1-\sigma}}{1-\sigma},\]

and a moderate degree of risk aversion by setting \(\sigma = 2\). The preference specification satisfies the assumption in proposition 5. We calibrate the fixed expenditures \(\tau\) that cannot be transferred to foreign lenders such that if the government is forced to repay debt in all contingencies, the marginally binding NBLs imply that the net foreign asset position of the country cannot fall below \(-100\%\) of average GDP in any productivity state.

\[^{30}\text{We thereby seek to capture the fact that industrialized countries, which default only rarely, virtually never have a net foreign asset position below \(-100\%\) of GDP, see figure 10 in Lane and Milesi-Ferretti (2007). It thus appears plausible to consider a calibration implying that countries cannot sustain higher external debt levels without running the risk of a sovereign default.}\]

It only remains to determine the default cost parameter \(\lambda\). While default (or contracting) costs are notoriously difficult to estimate, it is possible to exploit restrictions from our structural model to obtain an estimated lower bound for the costs of default. The idea underlying our estimation approach is to exploit the fact that default costs that accrue to the lender (but not those accruing to the borrower) can be estimated from financial market prices and information on default events. This is feasible because the borrower has to compensate the lender ex-ante for the default costs arising

\[^{29}\text{The quantitative results reported below are not very sensitive to the precise numbers used. A corresponding calibration at a quarterly frequency is employed in Adam (2011).}\]

\[^{30}\text{Average GDP is defined as the average output level associated with efficient investment, i.e., when } k_t = k^*(z_t) \text{ each period, as defined in proposition 4. We thereby average over the ergodic distribution of the } z \text{ process. For our parameterization this yields an average output level of 0.5647 before fixed expenditures. Furthermore, at the marginally binding NBLs, government decisions are exclusively determined by the desire to prevent debt from exploding, so that the marginally binding NBL cannot be used to calibrate the model. The resulting fixed or non-transferable expenditures are given by } \tau = 0.3540.\]

\[^{31}\text{Three out of the five industrialized countries approaching this boundary in the year 2004 later on faced fiscal solvency problems (Greece, Portugal and Iceland).}\]
on the lender’s side, so that lenders’ default costs are reflected in financial market prices and can thus be backed out from these.

To exploit this idea, we consider a slightly more general setting than that considered in the Ramsey problem (5) where the lender also bears default costs ($\lambda^l > 0$). Total default costs are then given by $\lambda = \lambda^l + \lambda^b$ with $\lambda^b$ denoting the borrower’s default cost. The structure of the Ramsey problem (5) then remains unchanged, except for the bond pricing equation (4), which has to be adapted so as to reflect the presence of the lenders’ default costs $\lambda^l$.

$$\frac{1}{1 + R(z_t, \Delta)} = \frac{1}{1 + r} \sum_{n=1}^{N} (1 - (1 + \lambda^l)^{\delta^n}) \cdot \pi(z^n|z_t)$$

(23)

Appendix A.9 shows how one can combine the previous equation with data on ex-post returns from Klingen, Weder, and Zettelmeyer (2004), who consider 21 countries over the period 1970-2000, and data on default events for the corresponding set of countries and years, kindly provided to us by Cruces and Trebesch (2011), to obtain an estimate for the lender’s default cost. This yields

$$\lambda^l = 6.1\%$$

and suggests that lenders suffer a loss of about 6% of the default amount in a sovereign default event. Note that this loss is in addition to the direct losses that result from incomplete repayment by the sovereign debtor: for every dollar that is not repaid to the lender, the lender suffers an additional loss of 6 cents.

The total costs of default include the costs accruing to the lender and to the borrower. Therefore, we consider in our quantitative analysis default cost levels $\lambda$ that exceed the estimated value of $\lambda^l$. Specifically, we shall consider default cost levels of 10% and 20%, respectively. A value of $\lambda = 10\%$ implies that about 60% of the overall default costs are born by the lender. A setting with $\lambda = 20\%$ is less conservative and implies that slightly more than 2/3 of the total default costs accrue on the borrower’s side.

### 5.2 Optimal Default with Business Cycle Shocks to Productivity

This section determines the optimal default policies for our calibrated model from the previous section. Determining the Ramsey optimal policies requires solving a non-linear stochastic dynamic optimization problem involving occasionally binding inequality constraints. Appendix A.10 explains how this has been achieved.

The top and bottom rows of figure 1 depict the optimal default policies for $\lambda = 10\%$ and $\lambda = 20\%$, respectively. Graphs on the left show the optimal default policy when

---

32 In the definition of the beginning of period wealth level (3), one also has to replace $\lambda$ by $\lambda^b$, where the latter captures the borrower’s proportional default costs.

33 Appendix A.8 shows that if a consumption allocation is feasible in a setting where default costs are born exclusively by the borrower, then it is also feasible if some of these costs are born by the lender, as long as the total amount of default costs $\lambda$ remains unchanged.
current productivity is high \((z^h)\), while graphs on the right depict the policy if current productivity is low \((z^l)\). Each graph of the figure reports the optimal default amount in the next period, if productivity in that period is low \((z^l)\). Default is optimally zero, whenever the high productivity state materializes tomorrow. To facilitate interpretation, the optimal default policies are depicted as a function of the net foreign asset position of the country, which is shown on the x-axis.\(^{34}\) Moreover, the default amounts (y-axis) and the net foreign asset positions (x-axis) are both normalized by average GDP. A value of \(-100\%\) on the x-axis, for example, corresponds to a situation where the government has issued repayment claims such that its net foreign asset position equals \(-100\%\) of average GDP. Since the net foreign asset position and beginning-of-period wealth commove positively in the optimal solution, the lowest value of the net foreign asset position for which policies are depicted in the graphs corresponds to the point where the country’s beginning-of-period wealth level has reached its marginally binding borrowing limit.

Figure 1 shows that default is more desirable, the lower is the country’s net foreign asset (or wealth) position and the lower are the default costs. Overall, however, default is suboptimal over a wide range of net foreign asset positions. For \(\lambda = 20\%\), for example, default is always suboptimal tomorrow, whenever today’s productivity is also low. If today’s productivity is high, then default is optimal tomorrow only if the net foreign asset position is close to its lowest level, i.e., if wealth is close to its marginally binding borrowing limit.

\(^{34}\)The net foreign asset position \(b_t\), defined in section 2.2.1, is a strictly increasing function of the state variable \(\tilde{w}_t\), allowing us to substitute \(\tilde{w}_t\) by \(b_t\) in the graphs when depicting optimal policies.
Given that for $\lambda = 0$ the optimal default policies are flat and strictly positively valued functions (this follows from proposition 4), figure 1 shows that already moderate levels of default costs cause default to be suboptimal over a large range of net foreign asset positions. The assumption of full repayment, standardly entertained in the Ramsey literature with incomplete markets, thus appears to provide a reasonable approximation to the fully optimal Ramsey policy for a wide range of net foreign asset positions.

Despite this fact, optimal default decisions have the potential to significantly increase welfare relative to an economy where default is rule out by assumption, especially if the country’s international wealth position is low. To illustrate this fact, let $c^1_t$ denote the optimal state contingent consumption path in an economy where default is ruled out by assumption, and $c^2_t$ the corresponding consumption path with (costly) optimal default decisions. We then compute the welfare equivalent consumption increase $\omega$, which causes the path $c^1_t$ to be welfare equivalent to path $c^2_t$ over the first 500 years\(^{35}\), which is given by

$$E_0 \left[ \sum_{t=0}^{500} \beta^t \left( \frac{(c^1_t + \omega(c^1_t + \bar{c}))^{1-\gamma}}{1 - \gamma} \right) \right] = E_0 \left[ \sum_{t=0}^{500} \beta^t \left( \frac{(c^2_t)^{1-\gamma}}{1 - \gamma} \right) \right], \quad (24)$$

where the expectations are evaluated by averaging over 10000 sample paths. Note that the consumption increase $\omega$ is measured against total domestic expenditures, which includes also the fixed expenditures $\bar{c}$. This leads to a potential understatement of the consumption equivalent welfare gain, but appears sensible given that the variable consumption component $c^1_t$ approaches zero as the beginning of period wealth level approaches the marginally binding natural borrowing limit.

Since default is never optimal for high wealth levels, the achievable welfare gain depend on the initial net foreign asset position. Lower wealth positions thereby give rise to higher welfare gains. Similarly, lower default costs also increase the achievable welfare gains. An upper bound for the welfare gains can thus be computed by considering a low value for the plausible range of default costs, say $\lambda = 10\%$, and a low value for the initial wealth position. The lowest sustainable wealth position, when default is ruled out, is the one for which the net foreign asset position reaches -100\% of GDP. At this point, the welfare gains are sizable and amount to $\omega = 1.9\%$ of total consumption expenditures each period when $\lambda = 10\%$. For somewhat higher initial wealth position, e.g. the one implying a net foreign asset position of -80\% of GDP, the welfare gains largely disappear: we then have $\omega = 0.01\%$.

To understand why welfare gains increase strongly as the initial net foreign asset position approaches its maximal level, consider figure 2. The figure depicts the optimal investment decisions for the case with optimal default and $\lambda = 10\%$ and for the case where default is simply ruled out. The figure depicts the optimal investment policies in deviation from the efficient investment level\(^{36}\) and as a function of

\(^{35}\)We choose a finite horizon to evaluate the welfare implications because the net foreign asset position is not necessarily stationary under the optimal policy.

\(^{36}\)The efficient investment level is the expected profit maximizing level defined in equation (14).
the country’s net foreign asset position. The figure shows that the investment margin in the no-default economy is strongly distorted as the net foreign asset position approaches $-100\%$ of GDP. Indeed, as proposition 5 shows: at the borrowing limit, optimal investment in the no default case is given by the one that would be efficient, if the lowest possible productivity state is expected to materialize for sure tomorrow. Given the persistence of the high and low productivity states, this explains why the investment distortion is larger in the high productivity state.

Figure 2 also reveals that the investment distortions are much smaller if default is optimally chosen. This occurs because the ability to default following low productivity realizations relaxes the marginally binding natural borrowing limits. This fact is illustrated in figure 3, which depicts - again for both productivity states - how the net foreign asset position at the marginally binding borrowing constraint depends on the level of default costs (depicted on the x-axis). For $\lambda = 100\%$, which is a setting where default is suboptimal or simply ruled out, the sustainable net foreign asset position equals $-100\%$, as implied by our calibration. For more reasonable default cost levels, say in the range of $10\%$ to $20\%$, the possibility to default strongly relaxes the sustainable net foreign asset position. This relaxation is due to a relaxation of the marginally binding natural borrowing limits and allows to carry more output risk and thus to sustain higher investment levels.\footnote{The two discontinuities in the net foreign asset positions present in figure 3 relate to two default cost thresholds, above which default in the low state tomorrow becomes suboptimal. Specifically, the first discontinuity arises at $\lambda = 19.23\%$, which equals one minus the transition probability from $z^1$ to $z^1$. Above this cost level, it becomes suboptimal to default in the next period, if current productivity is low. The second discontinuity arises at $\lambda = 80.77\%$, which equals one minus the probability of transiting from $z^6$ into $z^1$. Above this cost level, it also becomes suboptimal to default if the current state is high. For our calibrated model, the no-default assumption is thus fully optimal whenever $\lambda \geq 80.77\%$. More generally, the no-default assumption is consistent with optimality whenever default costs exceed a level of one minus the probability of reaching an undesirable state. In settings}
6 Optimal Default with Economic Disasters

The previous section showed that for plausible default cost levels it is suboptimal to default on government debt in a setting with business cycle shocks, provided the country is not too close to its borrowing limit. This section quantitatively evaluates to what extent this conclusion continues to be true in a setting with much larger economic shocks. Consideration of larger shocks is motivated by the observation that countries occasionally experience very large negative shocks, as previously argued by Rietz (1988) and Barro (2006), and that such shocks tend to be associated with a government default in the data. To capture the possibility of large shocks, we augment the model by including disaster like shocks to aggregate productivity and then explore the quantitative implications of disaster risk for optimal sovereign default decisions.

6.1 Calibrating Disasters

To capture economic disasters we introduce two disaster sized productivity levels to our aggregate productivity process. We add two disaster states rather than a single state to capture the idea that the size of economic disasters is uncertain ex-ante. The inclusion of two disaster states also allows us to calibrate the disaster shocks in a way that they match both the mean and the variance of GDP disaster events.

Using a sample of 157 GDP disasters, Barro and Jin (2011) report a mean reduction in GDP of 20.4% and a standard deviation of 12.64%. Assuming that it is equally likely to enter both disaster states from the ‘normal’ business cycle states \((z^h, z^l)\), one obtains the productivity levels \(z^d = 0.9224\) for a medium sized disaster and \(z^{dd} = 0.6696\) for a severe disaster. Our vector of possible productivity realizations where the probability of reaching an undesirable state is very low, default costs thus need to be close to the prohibitive level of 100% to make full repayment Ramsey optimal in all states.

\(^{38}\)Barro (2006) and Gourio (2010) also consider sovereign default in disaster states. Due to the different focus of their analysis, default probabilities and default rates are exogenous in their setting.
thus takes the form $Z = \{z^h, z^l, z^d, z^{dd}\}$, where the parameterization of the business cycle states $(z^h, z^l)$ is the same as in section 5.1.

The state transition matrix for the shock process is described by the matrix

$$
\pi = \begin{pmatrix}
0.7770 & 0.1850 & 0.019 & 0.019 \\
0.1850 & 0.7770 & 0.019 & 0.019 \\
0.1429 & 0.1429 & 0.3571 & 0.3571 \\
0.1429 & 0.1429 & 0.3571 & 0.3571 \\
\end{pmatrix},
$$

where the transition probability from the business cycle states into the disaster states is chosen to match the unconditional disaster probability of 3.8% per year, as reported in Barro and Jin (2011). We thereby assume that both disaster states are reached with equal likelihood. The persistence of the disaster states is set to match the average duration of GDP disasters, which equals 3.5 years. For simplicity we assume that conditional on staying in a disaster, the medium and severe disaster state are equally likely to materialize. Finally, the transition probabilities of the business cycle states are rescaled to reflect the transition probability into a disaster state.

We also recalibrate the subsistence level for consumption $\bar{c}$. As in section 5, we choose $\bar{c}$ such that in an economy where bonds must be repaid always, the economy can sustain a maximum net foreign debt position of 100% of average GDP in the business cycle states $(z^h, z^l)$.\(^{39}\)

### 6.2 Optimal Default with Disasters: Quantitative Analysis

Figure 4 depicts the optimal default policies for the economy with disaster shocks when $\lambda = 10\%$. Each panel in the figure corresponds to a different productivity state

\(^{39}\)This yields an adjusted value of $\bar{c} = 0.1899$. Choosing tighter limits does not affect the shape of the optimal default policies but only shifts the policies depicted in figure 4 ‘to the right’.

27
today and reports the optimal amount of default for tomorrow’s states \( z^l, z^d \) and \( z^{dd} \), respectively. As before, default is never optimal if the highest productivity state \( z^h \) realizes in the next period. Policies are depicted as a function of the country’s net foreign asset position (on the x-axis of each graph) and the default amounts (on the y-axis) are normalized by average GDP.

Figure 4 reveals that in the presence of large shocks, the introduction of optimal repayment decisions tremendously relaxes the marginally binding borrowing limits, much more than occurs in a setting with small shocks only. For the considered calibration, the sustainable net foreign asset position at the borrowing limit drops from -100% of GDP for the case where repayment must occur always to a level of about -1,000% of GDP. Figure 4 also shows that it is virtually never optimal to default in the low business cycle state \( (z^l) \), unless the net foreign asset position is very close to its maximally sustainable level, similar to section 5 where we considered business cycle shocks only. Furthermore, for a wide range of values for the net foreign asset position, it is optimal to default if the economy makes a transition from a business cycle state to a disaster state, see the top panels in the figure. Default is optimal for a transition to the severe disaster state \( (z^{dd}) \), even when the country’s net foreign asset position is positive before the disaster.\(^{40}\) The optimal default amount is thereby increasing as the country’s net foreign asset position worsens. Yet, once the economy is in a disaster state, a further default in the event that the economy remains in the disaster state is optimal only if the net foreign asset position is very low, see the bottom panels of figure 4. Since the likelihood of staying in a disaster state is quite high, choosing not to repay if the disaster persists would have large effects on interest rates ex-ante. As a result, serial default in case of a persistent disaster is only optimal if the net foreign asset position is sufficiently negative.

The overall shape of the optimal default policies is fairly robust to assuming different values for the default costs \( \lambda \). Larger default costs, for example, shift the default policies ‘towards the left’, i.e., default occurs only for more negative net foreign asset positions. Larger default costs also tighten the maximally sustainable net foreign asset positions, thereby reducing the range of net foreign asset positions over which optimal policies exist.

Figure 5 reports a typical sample path for the net foreign asset position and the amount of default induced by the optimal default policies shown in figure 4. The economy starts at a zero net foreign asset position and each model period corresponds to one year. The figure shows that it is optimal to improve the net foreign asset position when the economy is in the business cycle states, with faster improvements being optimal in the high business cycle state, see for example the 50 year period without economic disasters starting in year 130. Over this period the net foreign asset position improves from a value slightly below -100% of GDP to small positive values.

Figure 5 also shows that a transition to the severe disaster state usually leads to

\(^{40}\)With a positive net foreign asset position, default can occur because the country chooses to issue sufficient amounts of own debt, which are invested in the foreign bond, with the sole purpose to be able to default on domestic debt, if a transition to a severe disaster occurs.
Net Foreign Asset Dynamics under Optimal Default Policy (\(\lambda = 10\%\))

default, provided the transition does not occur via the medium disaster state first (as is the case for year 20). Also, following a disaster, the net foreign asset position deteriorates, whenever the disaster persists for more than one period (see for example year 40), otherwise the net foreign asset position is largely unaffected or improves even slightly (see year 86). This shows that for the considered level of default cost, default provides only partial insurance against disaster risk. As a result, the net foreign asset dynamics are typically characterized by rapid deteriorations during persistent disaster periods and gradual improvements during normal times, with the latter accelerating during high business cycles states.

7 Welfare Gains from Optimal Default

This section documents that optimal default gives rise to sizable welfare gains when the economy is subject to potential output disasters, as considered in section 6. Specifically, we determine the welfare equivalent consumption gains relative to a situation where repayment of debt is simply assumed to always occur, as defined in equation (24) in section 5.2. As we show, the welfare gains are surprisingly robust to the assumed level of default costs, as long as these do not reach prohibitive levels.

Figure 6 shows how the consumption equivalent welfare gains depend on the default costs (shown on the x-axis). The figure depicts these gains for three different initial wealth positions at time zero, which imply initial net foreign asset positions of 0\%, -40\% and -80\% of GDP, respectively. Depending on the initial wealth position, the welfare gains amount - for a broad range of default costs - to 0.5-1.5\% of consumption each period, with the welfare gains increasing further at low default
cost levels. Overall, the welfare gains are surprisingly robust over a wide range of default costs, instead prove more sensitive to the assumed initial net foreign asset position. For default costs $\lambda \geq 0.5$ the welfare gains from default decrease steeply and for $\lambda \geq 0.7$ welfare gains largely disappear. For such default cost levels it becomes suboptimal to insure against a future disaster state when the economy is already in a disaster, independently of the country’s net foreign asset position. As a result, the natural borrowing limits tighten significantly in the disaster states\(^{41}\) and the required amount of insurance in the business cycle state $(z^l, z^h)$ increases strongly as the net foreign asset position deteriorates. While some insurance via default is still optimal for such high cost levels, it does not give rise to sizable welfare gains as high default costs imply that insurance via default is also very costly.

### 8 Long Maturities and Optimal Bond Repurchase Programs

This section briefly discusses the effects of considering also bonds with longer maturities.

Introducing longer maturities of risk-free foreign debt has no consequences for the prior analysis, as such bond cannot be used in better ways to insure against domestic output risk. Like short-dated foreign bonds, they just provide a store of value. Issuing long-dated domestic bonds, however, can offer an advantage over issuing just short bonds, whenever the market value of long bonds reacts to domestic conditions in a way that allows the government to insure more efficiently against domestic shocks than a default on short-dated bonds. It would be desirable, for example, that the market value of outstanding long bonds decreases following a disaster shock, so that the government realizes a capital gain that lowers the overall debt burden. Such capital gains, however, do not materialize if repayment is assumed to occur in all

---

\(^{41}\)They reach the levels applying in the economy with non-defaultable bonds.
future states, as assumed in Angeletos (2002). The depreciation of the debt’s market value, can thus only be induced via the anticipation of future default on long-dated bonds.

Issuing long bonds will offer an advantage against the outright default on maturing short bonds, whenever the costs associated with repurchasing outstanding long bonds at the devaluated market price are lower than the costs induced by a default event on maturing short bonds. If both costs are identical, i.e., if the repurchase in a situation where default is anticipated in the future is associated with the same costs as a default on maturing bonds, then there will be of no additional value associated with issuing long bonds. The same holds true, if the costs of a repurchase are higher than those of an outright default. When these costs are lower, however, then there exists an advantage from issuing long-dated bonds. Consider, for example, the situation where a repurchase of long bonds does not give rise to any costs. The government will then find it optimal to fully insure domestic consumption, i.e., achieve the first best allocation, independently of the costs associated with an outright default on maturing bonds. The optimal bond issuance strategy will then have the feature that the government issues each period long bonds that (partially) default at maturity, depending on the productivity realization in some earlier period. The default at maturity needs to be calibrated such that the interim capital gains that realize fully insure domestic consumption against domestic productivity shocks, i.e., satisfies the proportionality restriction (15). The government can then repurchase the existing stock of long bonds, fully avoid default costs, and issue new long bonds with a new implicit contingent repayment profile, so as to insure against future shocks. In this way outright default on maturing bonds never occurs.

9 Conclusions and Outlook

In a setting with endogenously incomplete government bond markets, debt default is part of Ramsey optimal policy. Allowing the planner to choose whether or not to repay maturing debt relaxes the country’s borrowing limits, thereby increases international risk sharing and the efficiency of domestic investment. This leads to significant welfare gains, even when default costs are sizable.

Interestingly, the normative benchmark derived in the present analysis also has the potential to offer a plausible positive description of the actually observed debt and default patterns. The present setting with commitment has, for example, no difficulties in rationalizing the existing sovereign debt levels or in generating default patterns where default occurs following negative output realizations. The present approach may thus have an advantage relative to limited commitment approaches, which often face difficulties in explaining the existing sovereign debt levels\textsuperscript{42}, and struggle with the fact that default incentives are strongest following a positive innovation to domestic income, as the threat of a market exclusion is then least deterring\textsuperscript{43}. Exploring the

\textsuperscript{42}See, for example, the borrowing constraints in tables 2 and 3 reported in the quantitative analysis of Arellano and Heathcote (2010).

\textsuperscript{43}This feature prompts Arellano (2008), for example, to introduce state dependent default costs, which tilt the default profile towards low output states.
positive relevance of the presented normative theory of optimal default thus remains an interesting avenue for future research. This is especially true because the present analysis shows that Ramsey optimality is entirely consistent with a country choosing not to repay debt in a situation where it has sufficient resources to do so.

A Appendix

A.1 First Order Equilibrium Conditions

This appendix derives the necessary and sufficient first order conditions for problem (8). Using equation (7) to express beginning of period wealth, the problem is given by

$$\max_{\{b_t,a_t \geq 0,k_t \geq 0,\bar{c}_t \geq 0\}} E_0 \sum_{t=0}^{\infty} \beta^t u(\bar{c}_t)$$

s.t.:

$$\bar{c}_t = z_t \bar{k}_{t-1}^{\alpha} + b_{t-1} + (1 - \lambda)a_{t-1}(z_t) - \bar{c} - \bar{k}_t - \frac{1}{1+r}b_t - p'_t a_t$$

$$z_{t+1} \bar{k}_t^{\alpha} + b_t + (1 - \lambda)a_t(z_{t+1}) \geq NBL(z_{t+1}) \quad \forall z_{t+1} \in Z$$

$$\tilde{w}_0 = w_0, \quad z_0 \text{ given},$$

We now formulate the Lagrangian $\Lambda$, letting $\eta_t$ denote the multiplier on the budget constraint in period $t$, $\nu_t(z^n)$ the multiplier for the short-selling constraint on the Arrow security that pays off in state $z^n$ in $t+1$, and $\omega_t \in R^N$ the vector of multipliers associated with the natural borrowing limits for each possible realization of productivity in $t+1$, where $w_t(z^n)$ denotes the entry of the vector pertaining to productivity state $z^n$:

$$\Lambda = E_0 \left[ \sum_{t=0}^{\infty} \beta^t u(\bar{c}_t) \right]$$

$$+ \beta^t \eta_t \left( -\bar{c}_t + z_t \bar{k}_{t-1}^{\alpha} + b_{t-1} + (1 - \lambda)a_{t-1}(z_t) - \bar{c} - \bar{k}_t - \frac{1}{1+r}b_t - p'_t a_t \right)$$

$$+ \beta^t \sum_{i=1}^{N} \nu_t(z^n)a_t(z^n)$$

$$+ \beta^t \sum_{i=1}^{N} \omega_t(z^n) \left( z^n \bar{k}_t^{\alpha} + b_t + (1 - \lambda)a_t(z^n) - NBL(z^n) \right)$$

We drop the inequality constraints for $\bar{k}_t$ and $\bar{c}_t$, as the Inada conditions guarantee an interior solution for these variables whenever $\tilde{w}_0 > NBL(z^0)$.\footnote{This assumes that the fixed expenditures $\tau$ are truly fixed and accrue independently of the investment level, i.e., even if $k_t = 0$.} Differentiating the
Lagrangian with respect to the choice variables, one obtains

\[
\bar{c}_t : \quad u'(\bar{c}_t) - \eta_t = 0
\]

\[
b_t : \quad -\eta_t \frac{1}{1 + r} + \beta E_t \eta_{t+1} + \sum_{n=1}^{N} \omega_t(z^n) = 0
\]

\[
a_t(z^n) : \quad -\eta_t p_t(z^n) + \beta \pi(z^n | z_t) \eta_{t+1}(z^n)(1 - \lambda) + \nu_t(z^n) + \omega_t(z^n)(1 - \lambda) = 0 \quad \forall n \in N
\]

\[
\bar{k}_t : \quad -\eta_t + \alpha \bar{k}_t^{a-1} \beta E_t \eta_{t+1} z_{t+1} + \alpha \bar{k}_t^{a-1} \sum_{i=1}^{N} \omega_t(z^n) z^n = 0
\]

Using the FOC for consumption to replace \( \eta_t \) in the last three FOCs, deliver three Euler equations:

\[
\text{Bond:} -u'(\bar{c}_t) \frac{1}{1 + r} + \beta E_t u'(\bar{c}_{t+1}) + \sum_{n=1}^{N} \omega_t(z^n) = 0 \quad (26a)
\]

\[
\text{Arrow:} -u'(\bar{c}_t) p_t(z^n) + \beta \pi(z^n | z_t) u'(\bar{c}_{t+1}(z^n))(1 - \lambda) + \nu_t(z^n) + \omega_t(z^n)(1 - \lambda) = 0 \quad \forall n \in N \quad (26b)
\]

\[
\text{Capital:} -u'(\bar{c}_t) + \alpha \bar{k}_t^{a-1} \beta E_t u'(\bar{c}_{t+1}) z_{t+1} + \alpha \bar{k}_t^{a-1} \sum_{i=1}^{N} \omega_t(z^n) z^n = 0 \quad (26c)
\]

In addition, we have the following complementarity conditions \( \forall n \in N \):

\[
0 \leq a_t(z^n), \quad \nu_t(z^n) \geq 0, \quad \text{one holding strictly} \quad (26d)
\]

\[
0 \leq z^n \bar{k}_t^a + b_t + (1 - \lambda) a_t(z^n) - NBL(z^n), \quad \omega_{t+1}(z^n) \geq 0, \quad \text{one holding strictly}\quad (26e)
\]

Combined with the budget constraint, the Euler equations and the complementarity conditions constitute the necessary and sufficient optimality conditions for problem (8).

### A.2 Proof of Proposition 1

Consider some state contingent beginning-of-period wealth profile \( w_t \) arising from some combination of bond holdings, default decisions and capital investment \( (F_{t-1}, D_{t-1}, \Delta_{t-1}, k_{t-1}) \) in problem (5). We show below that one can generate the same state contingent beginning-of-period wealth profile \( \bar{w}_t = w_t \) in problem (8) by choosing \( \bar{k}_{t-1} = k_{t-1} \) and by choosing an appropriate investment profile \( (a_{t-1}, b_{t-1}) \). Moreover, the same amount of funds are required to purchase \( (a_{t-1}, b_{t-1}) \) in \( t - 1 \) than to purchase \( (F_{t-1}, D_{t-1}) \) when the default profile is \( \Delta_{t-1} \). With the costs of financial investments generating a particular future payout profile being the same in both problems, identical physical investments, and identical beginning of period wealth profiles, it then follows from constraints (5b) and (8b) that a consumption path that is feasible in (5) is also feasible in (8).

To simplify notation we establish the previous claim for the case with \( N = 2 \) productivity states only. The extension to more states is straightforward. Consider
the following state contingent initial wealth profile
\[
\begin{pmatrix}
  w_t(z^1) \\
  w_t(z^2)
\end{pmatrix} = \begin{pmatrix}
  z^1k_{t-1}^\alpha + F_{t-1} - D_{t-1}(1 - (1 - \lambda)\delta_{t-1}(z^1)) \\
  z^2k_{t-1}^\alpha + F_{t-1} - D_{t-1}(1 - (1 - \lambda)\delta_{t-1}(z^2))
\end{pmatrix}.
\]

As is easily verified, one can replicate this beginning-of-period wealth profile in problem (8) by choosing \( k_{t-1} = k_{t-1} \) and by choosing the portfolio

\[
\begin{align*}
  b_{t-1} &= F_{t-1} - D_{t-1}, \\
  a_{t-1} &= \begin{pmatrix}
  D_{t-1}\delta_{t-1}(z^1) \\
  D_{t-1}\delta_{t-1}(z^2)
\end{pmatrix}
\end{align*}
\]

The funds \( f_{t-1} \) required to purchase and issue \((F_{t-1}, D_{t-1})\) under the default profile \( \Delta_{t-1} = (\delta_{t-1}(z^1), \delta_{t-1}(z^2)) \) are given by

\[
f_{t-1} = \frac{1}{1 + r} F_{t-1} - \frac{1}{1 + R(z_{t-1}, \Delta_{t-1})} D_{t-1}
\]

where the interest rate satisfies

\[
\frac{1}{1 + R(z_{t-1}, \Delta_{t-1})} = \frac{1}{1 + r} \left( (1 - \delta_{t-1}(z^1))\pi(z^1 | z_{t-1}) + (1 - \delta_{t-1}(z^2))\pi(z^2 | z_{t-1}) \right). 
\]

The funds \( \tilde{f}_{t-1} \) required to purchase \((b_{t-1}, a_{t-1})\) are given by

\[
\tilde{f}_{t-1} = \frac{1}{1 + r} (F_{t-1} - D_{t-1}) + \frac{1}{1 + r} \left( \delta_{t-1}(z^1)\pi(z^1 | z_{t-1}) + \delta_{t-1}(z^2)\pi(z^2 | z_{t-1}) \right) G_{t-1}^N,
\]

where we used the price of the Arrow security in (6). As is easy to see \( \tilde{f}_{t-1} = f_{t-1} \), as claimed. This completes the proof that a consumption path that is feasible in (5) is also feasible in (8). Since \( a \geq 0 \), the reverse is also true, because equations (27) and (28) can then be solved for values \((D_{t-1}, F_{t-1}, \Delta_{t-1})\) satisfying \( D_{t-1} \geq 0 \), \( F_{t-1} \geq 0 \), and \( \Delta_{t-1} \in [0, 1]^N \), so that it is possible in problem (5) to obtain a portfolio with the same contingent payouts.\(^{45}\) Again, this portfolio has the same costs and thus admits for the same consumption choices. This completes the equivalence proof.

### A.3 Proof of Proposition 2

Let us define a critical future productivity state \( n^*_t \)

\[
n_t^* = \arg \max_{n \in \{[1, \ldots, N]\}} n
\]

\[
s.t. \sum_{i=n}^{N} \pi(z^j | z_t) \geq 1 - \lambda
\]

\(^{45}\)Note that for \( a < 0 \), no such choices would exist, which shows that \( a \geq 0 \) is required to obtain equivalence.
and choices

\[ \tilde{k}_t = \left( \alpha \beta \left( \frac{\sum_{n=1}^{n_t^*} \pi(z^n|z_t)z^{n_t^*} - \lambda}{1 - \lambda} + \sum_{n=n_t^*+1}^{T} \pi(z^n|z_t)z^n \right) \right)^{1/\lambda} \]  

\[ b_t = NBL(z^{n_t^*}) - z^{n_t^*} \tilde{k}_t^\alpha \]  

\[ c_t = 0 \]  

\[ a_t(z^n) = 0 \] for \( n \leq n_t^* \)  

\[ a_t(z^n) = \frac{NBL(z^n) - z^n \tilde{k}_t^\alpha - b_t}{(1 - \lambda)} > 0 \] for \( n > n_t^* \)

We now show that these choices satisfy the necessary and sufficient first order conditions of problem (11). Problem (11) can more explicitly be written as

\[
\begin{align*}
\min_{a_t, b_t, \tilde{k}_t, \tilde{c}_t} \quad & \bar{c} + \tilde{c}_t + \tilde{k}_t + \frac{1}{1 + r} b_t + \sum_{i=1}^{N} a_t(z^n) p_t(z^n) \\
\text{s.t.} \quad & z^{n_t^*} \tilde{k}_t^\alpha + b_t + (1 - \lambda)a_t(z^n) - NBL(z^n) \geq 0 \text{ for } n = 1, \ldots, N \\
& a_t(z^n) \geq 0 \text{ for } n = 1, \ldots, N \\
& \tilde{c}_t \geq 0
\end{align*}
\]

Obviously, it is optimal to choose \( \tilde{c}_t = 0 \), so we drop this variable from the problem from now own. Letting \( \omega_t(z^n) \) denote the Lagrange multipliers for the first set of constraints and \( \nu_t(z^n) \) the multipliers for the second set of constraints in (35), the first order necessary conditions are given by

\[ \tilde{k}_t : 1 + \alpha \tilde{k}_t^{\alpha - 1} \sum_{n=1}^{N} \omega_t(z^n)z^n = 0 \]  

\[ b_t : \frac{1}{1 + r} + \sum_{n=1}^{N} \omega_t(z^n) = 0 \]  

\[ a_t(z^n) : p_t(z^n) + \omega_t(z^n)(1 - \lambda) + \nu_t(z^n) = 0 \]

We also have the constraints

\[ z^{n_t^*} \tilde{k}_t^\alpha + b_t + (1 - \lambda)a_t(z^n) - NBL(z^n) \geq 0, \omega_t(z^n) \leq 0, \text{ one holding strictly} \]  

\[ a_t(z^n) \geq 0, \nu_t(z^n) \leq 0, \text{ one holding strictly} \]

Conditions (38) and (40) can equivalently be summarized as

\[ p_t(z^n) + \omega_t(z^n)(1 - \lambda) \geq 0, a_t(z^n) \geq 0, \text{ one holding strictly} \]

so that the first order conditions are given by (36),(37), (39) and (41). Since the objective is linear and the constraint set convex, the first order conditions are necessary
and sufficient. We now show that the postulated solution satisfies these first order conditions.

Since \( a(z^n) > 0 \) for \( n > n_t^* \), equation (41) implies

\[
\omega_t(z^n) = -\frac{p_t(z^n)}{1 - \lambda} = -\frac{1}{1 + r} \frac{\pi(z^n|z_t)}{1 - \lambda} \quad \text{for all } n > n_t^*
\]

We now conjecture (and verify later) that

\[
\omega_t(z^n) = 0 \quad \text{for all } n < n_t^* \quad \text{(42)}
\]

\[
\omega_t(z_n^*) = -\frac{1}{1 + r} - \sum_{n=n^*+1}^{N} \omega_t(z^n) \quad \text{(43)}
\]

We then get from the optimal investment level from (36)

\[
\tilde{k}_t = \left( -\alpha \sum_{n=n_t^*}^{N} \omega_t(z^n) z^n \right) \frac{1}{1 + \alpha}
\]

\[
= \left( \alpha \beta \left( \sum_{n=n_t^*}^{N} \frac{\pi(z^n|z_t)}{1 - \lambda} z^{n^*} + \frac{1}{1 + r} \sum_{n=n_t^*+1}^{N} \pi(z^n|z_t) z^n \right) \right) \frac{1}{1 + \alpha}
\]

Since

\[
\sum_{n=n_t^*+1}^{N} \omega_t(z^n) = \frac{1}{1 + r} \sum_{n=n_t^*+1}^{N} \frac{\pi(z^n|z_t)}{1 - \lambda} < \frac{1}{1 + r}
\]

we have that \( \omega_t(z_n^*) < 0 \) as conjectured in (43). We obtain from (39) using \( a(z^{n*}) = 0 \) the optimal bond holdings

\[
b_t = NBL(z^{n*}) - z^{n*} \tilde{k}_t^a
\]

The optimal choice \( a(z^n) \) for \( n > n_t^* \) then follows from (39):

\[
a_t(z^n) = \frac{NBL(z^n) - z^n \tilde{k}_t^a - b_t}{1 - \lambda}
\]

As is easily verified (36), (37) and (41) hold for the conjectured solution. The second inequality in (39) also holds. The first inequality holds in (39) holds by construction for \( n \geq n_t^* \). For \( n < n_t^* \) it holds because \( NBL(z^n) \) is non-decreasing in \( n \) due to (12), output \( z^n \tilde{k}_t^a \) is strictly increasing, and the first inequality holds with equality for \( n = n_t^* \). As a result, the first inequality in (39) holds strictly for \( n < n_t^* \), justifying our conjecture (42). This proves that all first order conditions hold for the conjectured solution (30)-(34).

Since the solution (30)-(34) to (11) is at most linear in the NBLs showing up in the constraints of (11), the minimized objective is also a linear function of these NBLs. The fixed point defined by (11) is thus characterized by a system of equations that is linear, which generically admits for a unique solution.
A.4 Proof of Proposition 3

We start by proving point 2 in proposition 3. Suppose that in some productivity state $z^n$ and some period $t$, beginning-of-period wealth falls short of the limits implied by the marginally binding NBL computed in appendix A.3, i.e.,

$$\tilde{w}_t(z^n) = NBL(z^n) - \varepsilon,$$

(44)

for some $\varepsilon > 0$. We then prove below that for at least one contingency $z^j$ in $t + 1$, which can be reached from $z^n$ in $t$ with positive probability, it must hold that

$$\tilde{w}_{t+1}(z^j) \leq NBL(z^j) - \varepsilon(1 + r),$$

(45)

so that along this contingency the distance to the marginally binding NBL is increasing at the rate $1 + r > 1$ per period. Since the same reasoning applies also for future periods, and since the marginally binding NBLs assume finite values, this implies the existence of a path of productivity realizations along which future wealth far in the future becomes unboundedly negative, so that any finite natural borrowing limit will be violated with positive probability.

It remains to prove that if (44) holds in period $t$ and contingency $z^n$, this implies that (45) holds for some contingency $z^j$ in $t + 1$ and that $z^j$ can be reached from $z^n$ with positive probability. Suppose for contradiction that

$$\tilde{w}_{t+1}(z^h) > NBL(z^h) - \varepsilon(1 + r)$$

(46)

for all $h$ that can be reached from $z^n$, i.e., for which $\pi(z^h|z^n) > 0$. The cost minimizing way to satisfy the constraints (46) for all $h$, when replacing the strict inequality by a weak one, is to choose the solution (30), (32)–(34) and $b_t = NBL(z^n) - z^n \tilde{k}_t - \varepsilon$. Achieving this requires $NBL(z^n) - \varepsilon$ units of funds in $t$, which in turn implies that satisfying constraints (46) with strict inequality requires strictly more funds than are available in $t$. As a result, (45) must hold for at least one $j$ that can be reached from $z^n$ with positive probability. This concludes the proof of point 2 in the proposition.

The proof of point 1 is relatively straightforward. If $\tilde{w}_t(z^n) \geq NBL(z^n)$, then it is feasible to choose the policies (30), (31), (33), (34) and a non-negative consumption level in period $t$. This will give rise to beginning-of-period wealth levels in the future that are equal or above the marginally binding NBLs. Since the latter assume finite values, this proves the existence of a policy with non-explosive debt dynamics.

A.5 Proof of Proposition 4

We first show that the proposed consumption solution (13) and investment policy (14) satisfy the budget constraint, that the inequality constraints $a \geq 0$ are not binding, and that the NBLs are also not binding. Thereafter, we show that the remaining first order conditions of problem (8), as derived in appendix A.1, hold.

We start by showing that the portfolio implementing (13) in period $t = 1$ is consistent with the flow budget constraint and $a \geq 0$. The result for subsequent

46To verify this just solve the minimization problem (35) where the constraints on future wealth are replaced by (46) with the strict inequality replaced by a weak one.
periods follows by induction. In period \( t = 1 \) with productivity state \( z^n \), beginning-of-period wealth under the investment policy (14) is given by

\[
\tilde{w}_1^n = z^n (k^*(z_0))^\alpha + b_0 + a_0(z^n)
\]  

(47)

To insure that consumption can stay constant from \( t = 1 \) onwards we need again

\[
\tilde{c} = (1 - \beta)(\Pi(z^n) + \tilde{w}_n^n)
\]  

(48)

for all possible productivity realizations \( n = 1, \ldots, N \). This provides \( N \) conditions that can be used to determine the \( N + 1 \) variables \( b_0 \) and \( a_0(z^n) \) for \( n = 1, \ldots, N \). We also have the condition \( a_0(z^n) \geq 0 \) for all \( n \) and by choosing \( \min_n a_0(z^n) = 0 \), we get one more condition that allows to pin down a unique portfolio \((b_0, a_0)\). Note that the inequality constraints on \( a \) do not bind for the portfolio choice, as we have one degree of freedom, implying that the multipliers \( v_1(z^n) \) in appendix A.1 are zero for all \( n = 1, \ldots, N \). It remains to show that the portfolio achieving (48) is feasible given the initial wealth \( \tilde{w}_0 \). Using (47) to substitute \( \tilde{w}_n^n \) in equation (48) we get

\[
\tilde{c} = (1 - \beta)(\Pi(z^n) + z^n (k^*(z_0))^\alpha + b_0 + a_0(z^n)) \quad \forall n = 1, \ldots, N.
\]

Combining with (13) we get

\[
\Pi(z^n) + z^n (k^*(z_0))^\alpha + b_0 + a_0(z^n) = \Pi(z_0) + \tilde{w}_0
\]

Multiplying the previous equation with \( \pi(z^n|z_0) \) and summing over all \( n \) one obtains

\[
E_0 [\Pi(z_1) + z_1 (k^*(z_0))^\alpha] + b_0 + \sum_{n=1}^{N} \pi(z^n|z_0) a_0(z^n) = \Pi(z_0) + \tilde{w}_0.
\]

Using \( \Pi(z_0) = -k^*(z_0) + \beta E_0 [z_1 (k^*(z_1))^\alpha] - \tilde{c} + \beta E_0 [\Pi(z_1)] \) and (6) the previous equation delivers

\[
(1 - \beta) E_0 [\Pi(z_1) + z_1 (k^*(z_0))^\alpha] + b_0 + (1 + r)p_0' a_0 = -k^*(z_0) + \tilde{w}_0 - \tilde{c}
\]

Using \( \beta = 1/(1 + r) \) this can be written as

\[
(1 - \beta) \left( E_0 [\Pi(z_1) + z_1 (k^*(z_0))^\alpha] + \frac{1}{\beta} p_0 a_0 + b_0 \right) + \frac{1}{1 + r} b_0 + p_0' a_0 = -k^*(z_0) + \tilde{w}_0 \quad (49)
\]

From substituting (47) into (48), multiplying the result with \( \pi(z^n|z_0) \) and summing over all \( n \), it follows that the terms in the first line of the previous equation are equal to

\[
(1 - \beta) \left( E_0 [\Pi(z_1) + z_1 (k^*(z_0))^\alpha] + \frac{1}{\beta} p_0' a_0 + b_0 \right) = \tilde{c} - \tilde{c}
\]

where we also used (6) and \( 1 + r = 1/\beta \). Using this result to substitute the first line in (49) shows that (49) is just the flow budget equation for period zero. This proves
that the portfolio giving rise to (48) in \( t = 1 \) for all \( n = 1, ..., N \) satisfies the budget constraint of period \( t = 0 \). The results for \( t \geq 1 \) follow by induction. The result (15) follows from substituting (47) into (48) and noting that \( b_0 \) is not state contingent.

From equation (48) and the fact that \( (z^n_t) \) is bounded, it follows that \( \tilde{w}_t \) is bounded so that the process for beginning-of-period wealth automatically satisfies the marginally binding NBLs. The multipliers \( \omega_{t+1} \) in appendix A.1 are thus equal to zero for all \( t \) and all contingencies. Using \( v_t(z^n) \equiv 0, \omega_{t+1} \equiv 0 \), the fact that capital investment is given by (14) and that the Arrow security price is (6), the Euler conditions (26a) - (26c) all hold when consumption is given by (13). This completes the proof.

A.6 Proof of proposition 5

Since we start with a beginning-of-period wealth level at the marginally binding NBL, we necessarily have \( \tilde{e}_t = 0 \) (otherwise one could afford an even lower initial wealth level and satisfy all constraints, which would be inconsistent with the definition of the marginally binding NBLs given in (11)). Indeed, the available beginning of period wealth \( \tilde{w}_t \) is just enough to insure that \( \tilde{w}_{t+1} \geq NBL(z_{t+1}) \) for all possible future productivity states \( z_{t+1} \). The optimal choices \( a_t \in R^N, b_t, \tilde{k}_t \) are thus given by the cost-minimizing choices satisfying \( a_t \geq 0 \) plus the marginally binding NBLs in \( t+1 \) for all possible future productivity states. Formally,

\[
\begin{align*}
\min_{a_t, b_t, \tilde{k}_t} \tilde{k}_t + \frac{1}{1+r} b_t + \sum_{i=1}^{N} a_t(z^n) p_t(z^n) \\
\text{s.t.} \quad z^n \tilde{k}_t^\alpha + b_t + (1 - \lambda) a_t(z^n) - NBL(z^n) &\geq 0 \quad \text{for } n = 1, ..., N \\
a_t(z^n) &\geq 0 \quad \text{for } n = 1, ..., N
\end{align*}
\]

The optimal choices are thus equivalent to those defining the NBLs in Appendix A.3, see problem (35). As shown in that appendix, the optimal choices are given by (30)-(34) provided (12) holds.

A.7 Proof of proposition 6

Using the assumed policies, \( \frac{1}{1+\tau} = \beta \) and \( p_t(z^n) = \frac{\pi(z^n|z_t)}{1+r} \), the Euler equations (26a)-(26c) for \( j = 0 \) imply

\[
\begin{align*}
u_t'(\tilde{c}_t) &= E_t u'(\tilde{c}_{t+1}) \\
u_t(z^n) &= \beta \pi(z^n|z_t) (u'(\tilde{c}_t) - u'(\tilde{c}_{t+1}(z^n))(1 - \lambda)) \quad \forall n \in N \\
0 &= -u'(\tilde{c}_t) + \alpha \tilde{k}_t^{\alpha-1} \beta E_t u'(\tilde{c}_{t+1}) z_{t+1}
\end{align*}
\]

We show below that the Euler equation errors for \( j = 0 \) converge to zero and that \( v_t(z^n_t) \geq 0 \) as the wealth \( \tilde{w}_t \) increases without bound. Under the assumed policies
wealth evolves according to

$$
\tilde{w}_{t+1} = z_{t+1} k^*(z_t) + b_t
$$

$$
= z_{t+1} k^*(z_t) + \frac{1}{\beta} (\tilde{w}_t - k^*(z_t) - (1 - \beta)(\Pi(z_t) + \tilde{w}_t) - \tilde{c})
$$

$$
= \tilde{w}_t + z_{t+1} k^*(z_t) - \frac{1}{\beta} k^*(z_t) - \frac{(1 - \beta)}{\beta} (\Pi(z_t)) - \frac{1}{\beta} \tilde{c}
$$

$$
= \tilde{w}_t + \Pi(z_t) + z_{t+1} k^*(z_t) - \frac{1}{\beta} k^*(z_t) - \frac{1}{\beta} (\Pi(z_t)) - \frac{1}{\beta} \tilde{c}
$$

$$
= \tilde{w}_t + \Pi(z_t) - E_t[\Pi(z_{t+1})]
$$

Since the fluctuations in the last two terms are bounded, the fluctuations in wealth for a finite period will be bounded as well. We now show that one can find for \( j = 0 \) a wealth level that implies a residual below \( \epsilon \). Using the boundedness of the fluctuations in wealth we will then be able to define a wealth level which ensures that for a finite number of subsequent periods the residuals are also below \( \epsilon \). Using the assumed consumption policy and the previous result, we have that

$$
E_t[\tilde{c}_{t+1}] = (1 - \beta) E_t[(\Pi(z_t) + \tilde{w}_t)]
$$

$$
= (1 - \beta) (E_t[\Pi(z_{t+1})] + \tilde{w}_t + \Pi(z_t) - E_t[\Pi(z_{t+1})])
$$

$$
= (1 - \beta) (\tilde{w}_t + \Pi(z_t))
$$

i.e. consumption follows a random walk. Now consider equation (50a), which requires

$$
u'(\tilde{c}_t) = E_t u'(\tilde{c}_{t+1})
$$

$$
= \sum_{n=1}^{N} \pi(z^n) u'(\tilde{c}_{t+1}(z^n))
$$

(51)

From the intermediate value theorem we have

$$
u'(\tilde{c}_{t+1}(z^n)) = u'(\tilde{c}_t) + u''(\tilde{c}_t)(c^n - \tilde{c}_t)
$$

where \( c^n \) is chosen from the bounded interval \([\min_n u'(\tilde{c}_{t+1}(z^n)), \max_n u'(\tilde{c}_{t+1}(z^n))]\), whose width is invariant to \( \tilde{w}_t \). Using the previous result, we can rewrite (51) as

$$
1 = \sum_{n=1}^{N} \pi(z^n) \left( 1 + \frac{u''(\tilde{c}_t)}{u'(\tilde{c}_t)}(c^n - \tilde{c}_t) \right)
$$

There is thus an Euler equation residual of size

$$
\frac{u''(\tilde{c}_t)}{u'(\tilde{c}_t)} \sum_{n=1}^{N} \pi(z^n)(c^n - \tilde{c}_t)
$$
For our policies the difference \(c^* - \tilde{c}_t\) is bounded and invariant to wealth. Moreover, \(u''(\tilde{c}_t)/u'(\tilde{c}_t) \to 0\) for large wealth levels under the stated consumption policy. This implies that for any given \(\epsilon > 0\) we can find a wealth level \(w^*\) so that the residual falls below \(\epsilon\). As the fluctuations in wealth are bounded for a fixed horizon \(T\) and do not depend on the wealth level, we can therefore find a wealth level \(\tilde{w}\) that ensures that for all possible fluctuations the wealth level stays above \(w^*\). This implies that for all \(j = 0, ..., T - 1\) the Euler equation errors are sufficiently small.

Similar arguments can be made to show that (50c) holds and that (50b) implies \(v_t(z^n) \geq 0\) for \(\lambda > 0\) and a sufficiently large wealth level. We omit the proof here for sake of brevity.

### A.8 Default Costs Born by Lender

This appendix shows that if an allocation is feasible in a setting in which the settlement costs associated with a default are born by the borrower, then it is also feasible in a setting in which some or all of these costs are born by the lender. For simplicity, we only consider the extreme alternative where all costs are born by the lender. Intermediate cases can be covered at the costs of some more cumbersome notation.

Consider a feasible choice \(\{G_L^t \geq 0, G_S^t \geq 0, \Delta_t \in [0, 1]^N, k_t \geq 0, c_t \geq \tilde{c}_t\} \subseteq \mathfrak{C}\), i.e., a choice that satisfies the constraints of the government’s problem (5), which assumes \(\lambda^l = 0\) and \(\lambda^b = \lambda\). Let variables with a tilde denote the corresponding choices in a setting in which the lender bears the settlement costs, i.e., where \(\lambda^l = \lambda\) and \(\lambda^b = 0\). We show below that it is then feasible to choose the same real allocation, i.e., to choose \(\tilde{k}_t = k_t\) and \(\tilde{c}_t = c_t\), provided one selects appropriate values for \(G_L^t\), \(G_S^t\) and \(\Delta_t\).

First, note that in a setting where foreign investors bear the settlement costs, the discount rate \(\tilde{R}(z_t, \Delta_t)\) on domestic bonds satisfies

\[
1 + r = \frac{1 - (1 + \lambda) \sum_{n=1}^{N} \tilde{\delta}_{t}^{n} \Pi(z^n | z_t)}{1 + R(z_t, \Delta_t)}
\]

where the denominator on the right-hand side denotes the issuance price of the bond and the numerator the expected repayment net of the lender’s settlement cost. The previous equation thus equates the expected returns of the domestic bonds with the expected return on the foreign bond.
Next, consider the following financial policies:\footnote{Lengthy but straightforward calculations, which are available upon request, show that these policies satisfy $\Delta_t \in [0,1]^N$, although they may imply $\tilde{G}_t^L < 0$, which requires the government to issue also safe bonds, i.e., bonds that promise full repayment in the explicit and implicit component of their contract.}

\begin{align}
\tilde{\Delta}_t &= (1 - \lambda)\Delta_t \frac{G_t^S}{G_t^L} \\
\tilde{G}_t^S &= \frac{1 + \tilde{R}(z_t, \tilde{\Delta}_t)}{R(z_t, \Delta_t)} \frac{R(z_t, \Delta_t)}{1 + R(z_t, \Delta_t)} G_t^S \\
\tilde{G}_t^L &= G_t^L + \tilde{G}_t^S - G_t^S
\end{align}

As we show below, in a setting in which settlement costs are born by the lender, the financial policies $\{\tilde{G}_t^L, \tilde{G}_t^S, \tilde{\Delta}_t\}_{t=0}^\infty$ give rise to the same state-contingent financial payoffs as generated by the policies $\{G_t^L, G_t^S, \Delta_t\}_{t=0}^\infty$ in a setting in which settlement cost are born by the borrower. Therefore, as claimed, the former policies allow to implement the same real allocations.

Consider the financial flows generated by the policy component $(\tilde{G}_t^L, \tilde{G}_t^S, \tilde{\Delta}_t)$. In period $t$, the financial inflows are given by

$$\frac{\tilde{G}_t^S}{1 + R(z_t, \Delta_t)} - \tilde{G}_t^L$$

Using the definitions (53), it is straightforward to show that these inflows are equal to

$$\frac{1}{1 + R(z_t, \Delta_t)}G_t^S - G_t^L$$

which are the inflows under the policy $(G_t^L, G_t^S, \Delta_t)$ in a setting where settlement costs are born by the lender.

We show next, that the financial flows in $t + 1$ are also identical under the two policies. The financial inflows generated by the policy choices $(\tilde{G}_t^L, \tilde{G}_t^S, \tilde{\Delta}_t)$ in some future contingency $n \in \{1, \ldots, N\}$ in period $t + 1$ are given by

$$-\tilde{G}_t^S(1 - \delta_t^n) + \tilde{G}_t^L$$

From the first and last equation in (53), we obtain that these flows are equal to

$$-(1 - (1 - \lambda)\delta_t^n)G_t^S + G_t^L$$

which are the inflows generated by the policy $(G_t^L, G_t^S, \Delta_t)$ in a setting where settlement cost are born by the lender, which completes the proof.
A.9 Estimation of Lender’s Default Costs

Consider a non-contingent one period bond that in explicit legal terms promises to repay one unit and that has an associated implicit default profile \( \Delta = (\delta^1, ..., \delta^n) \in [0, 1]^n \). A risk-neutral foreign lender, who bears proportional default costs \( \lambda^b \) in the event of default and can earn the gross return \( 1 + r \) on alternative investments, will price this bond according to equation (23). The asset pricing equation (23) can be used to to obtain an estimate for \( \lambda^l \).

First, we define the ex-post return \( epr_t \) on a government bond

\[
1 - \sum_{n=1}^{N} \delta^n \Pi(z^n | z_t) \\
1 + epr_t = \frac{1}{1 + R(z_t, \Delta)}
\]

which is the bond return net of the loss due to non-repayment. Ex-post returns can be measured from financial market data. Using the previous equation to substitute \( \sum_{n=1}^{N} (1 - \delta^n) \cdot \pi(z^n | z_t) \) on the r.h.s. of equation (23) and applying the unconditional expectations operator\(^{48}\), one obtains

\[
\lambda^l = \frac{E[epr_t - r]}{E[1 + R(z_t, \Delta) \sum_{n=1}^{N} \delta^n \Pi(z^n | z_t)]}.
\]

Information about the average excess return, which shows up in the numerator of the previous equation, can be obtained from Klingens, Weder, and Zettelmeyer (2004), who consider 21 emerging market economies over the period 1970-2000. Using data from table 3 in Klingens, Weder, and Zettelmeyer (2004), the average excess return varies between -0.2% and +0.5% for publicly guaranteed debt, depending on the estimation method used.\(^{49,50}\) We use the average of the estimated values and set \( E[epr_t - r] = 0.15\% \).

We now turn to the denominator on the r.h.s. of equation (54). Using a first order approximation we obtain

\[
E\left[ (1 + R(z_t, \Delta)) \sum_{n=1}^{N} \delta^n \Pi(z^n | z_t) \right] \approx E\left[ 1 + R(z_t, \Delta) \right] E\left[ \sum_{n=1}^{N} \delta^n \Pi(z^n | z_t) \right],
\]

\(^{48}\)The expectations operator integrates over the set of possible histories \( z^t = (z_t, z_{t-1}, ...) \).

\(^{49}\)As suggested in Klingens, Weder, and Zettelmeyer (2004), we use the return on a 3 year US government debt instrument as the safe asset, since it approximately has the same maturity as the considered emerging market debt.

\(^{50}\)The fact that ex-post excess returns on risky sovereign debt are relatively small or sometimes even negative is confirmed by data provided in Eichengreen and Portes (1986) who compute ex-post excess returns using interwar data. The negative ex-post excess returns likely arise due to the presence of sampling uncertainty: the high volatility of the nominal exchange rate makes it difficult to estimate the mean ex-post excess returns.
where the last term equals (again to a first order approximation)

\[ E \left[ \sum_{n=1}^{N} \delta^n \Pi(z^n|z_t) \right] \approx \Pr(\delta > 0)E[\delta|\delta > 0]. \]

Using data compiled by Cruces and Trebesch (2011), who kindly provided us with the required information, we observe for the 21 countries considered in Klingen et al. (2004) and for the period 1970-2000 a total of 58 default events, so that the average yearly default probability equals 8.9%. The average haircut conditional on a default was 25%, so that these figures imply

\[ E \left[ \sum_{n=1}^{N} \delta^n \Pi(z^n|z_t) \right] \approx 2.22%. \]

The average ex-ante interest rate \( R(z_t, \Delta) \) appearing in equation (55) can be computed by adding to the average ex-post return of 8.8% reported in table 3 in Klingen, Weder, and Zettelmeyer (2004) for publicly guaranteed debt, the average loss due to default, which equals 2.22% to first order, so that \( R(z_t, \Delta) \approx 11.02\% \). Combining these results to evaluate \( \lambda^I \) in equation (54) delivers our estimate value for the default costs accruing to lenders.

A.10 Numerical Solution Approach

We solve the recursive version of the Ramsey problem from section 2.2.3 using global solution methods, so as to account for the non-linear nature of the optimal policies. The state space \( S \) of the problem is given by

\[ S = \{ z^1 \times [NBL(z^1), w_{\text{max}}], \ldots, z^N \times [NBL(z^N), w_{\text{max}}] \} \]

where \( NBL(z^n) \) denotes the marginally binding natural borrowing limits and \( w_{\text{max}} \) is chosen such that in equilibrium optimal policies never imply wealth values above this threshold.

We want to describe equilibrium in terms of time-invariant policy functions that map the current state into current policies. Hence, we want to compute policies

\[ \tilde{f} : (z_t, w_t) \rightarrow (\{c_t, k_t, b_t, a_t\}), \]

where their values (approximately) satisfy the optimality conditions derived in A.1. We use a time iteration algorithm where equilibrium policy functions are approximated iteratively. In a time iteration procedure, one takes tomorrow’s policy (denoted by \( f^{\text{next}} \)) as given and solves for the optimal policy \( f \) today, which in turn is used to update the guess for tomorrow’s policy. Convergence is achieved once \( ||f - f^{\text{next}}|| < \epsilon \), where we set \( \epsilon = 10^{-5} \). We then set \( \tilde{f} = f \). In each time iteration step we solve for optimal policies on a sufficient number of grid points distributed over the continuous part of the state space. Between grid points we use linear splines to interpolate tomorrow’s policy. Following Garcia and Zangwill (1981), we can transform the complementarity conditions of our first order equilibrium conditions into
equations. For more details on the time iteration procedure and how one transforms complementarity conditions into equations, see for example, Brumm and Grill (2010). To come up with a starting guess for the consumption policy we use the fact that at the NBLs optimal consumption equals the subsistence level. We therefore guess a convex, monotonically increasing function $g$ which satisfies $g(z^i, NBL(z^i)) = \bar{c} \forall i$ and use a reasonable guess for $g(z^i, w_{\text{max}})$.

References


