Financial Networks and Contagion

Matthew Elliott∗ Benjamin Golub† Matthew O. Jackson‡

Revision: November 2012

Abstract

We model financial contagions and cascades of defaults among organizations that have a network of cross holdings. We first identify a network-based measure that captures the impact of changes in one organization’s value on other organizations’ values. We use the measure to study both integration (the increasing of cross holdings) and diversification (the spreading out of cross holdings). We show that diversification initially increases the probability and extent of cascades as a network of interdependencies grows, and eventually the probability and extent of cascades decreases once organizations become less tied to specific other organizations. Integration also faces tradeoffs: increased dependence on other organizations versus less sensitivity to own investments. We briefly discuss incentives to seek bailouts, and associated moral-hazard issues. We also show that once an organization approaches a bankruptcy threshold, there are no trades of cross holdings or assets at fair prices that can lower the probability of its failure, and that unduly favorable trades for that organization and/or a direct injection of capital are necessitated. Finally, we illustrate some aspects of the model with European debt cross holdings.

Keywords: financial networks, networks, contagion, cascades, financial crises, bankruptcy, diversification, integration, globalization

JEL Classification Numbers: G32, D85, G01, F36, G33, G38, F15

1 Introduction

Along with increasing "globalization" comes a web of increased interdependencies among organizations throughout the world, from central banks and governments, to private banks and investment banks, to firms, and even private investors and consumers. These interconnections can lead to cascading defaults and failures. Such cascades are often avoided by

∗HSS, Caltech. Email: melliott@caltech.edu,
†Department of Economics, MIT. Email: bgolub@mit.edu, web site: http://www.stanford.edu/~bgolub
‡Department of Economics, Stanford University, the Santa Fe Institute, and CIFAR. Email: jacksonm@stanford.edu, web site: http://www.stanford.edu/~jacksonm. Jackson gratefully acknowledges financial support from NSF grant SES-0961481 and grant FA9550-12-01-0411 from AFOSR and DARPA. All authors thank Microsoft Research New England Lab for research support.
massive bailouts of institutions deemed “too-big-to-fail”, such as interventions in cases such as A.I.G., Fannie Mae, Freddie Mac, and General Motors; or in cases such as the European Commission’s interventions in Greece and Spain. Although such bailouts circumvent the widespread failures that were more prevalent in the nineteenth and early twentieth centuries, they re-emphasize the need to understand the network risks to better understand how such cascades can be defused long before they happen. Understanding exactly how cascades relate to the network of financial interdependencies is thus essential.

In this paper we develop a general model that produces new insights regarding financial contagions and cascades of defaults among a network of organizations whose values are interrelated through cross holdings. In our model, cross holdings of organizations’ debts, as well as exchanges of assets (through, for example, various swaps), can lead to discontinuous changes in values and cascading defaults when one organization hits a liquidity or bankruptcy constraint. Relatively small and even organization-specific shocks can be greatly amplified through such networks.\(^1\)

In our model, organizations can invest in primitive assets (any factors of production or other investments) as well as in each other.\(^2\) Cross holdings in each other lead to a well-known problem of inflating book values, and so we begin our analysis by deriving an expression for the true “market value” of organizations and in particular showing how each organization’s market value depends on the values of the primitive assets and any liquidity/bankruptcy shocks that have hit the economy. In particular, the impact of the interlinked bankruptcy constraints is captured via a network. We develop a new network-based measure that captures the impact of changes in one organization’s values on other organizations’ values. An implication of failures being complementary is that cascades occur in “waves”. Some initial failures are alone enough to cause a second “wave” of organizations to fail. Once these organizations fail, a third “wave” of failures may occur, and so on. We show that our network measure determines the extent of cascades of failure and the amplification of shocks. We also provide a simple algorithm for identifying the cascades.

With this methodology in hand, our main results show how the probability of cascades and their extent depend on two key aspects of cross holdings: integration and diversification. Integration refers to the level of exposure of organizations to each other: how much of an organization is privately held by final investors, and how much is cross held by other

---

\(^1\) The discontinuities incurred when an organization fails can include the cost of liquidating assets, the (temporary) misallocation of productive resources, as well as direct legal and administrative costs. Given that efficient investment or production can involve a variety of synergies and complementarities, any interruption in the ability to invest or pay for and acquire some factors of production can lead to discontinuously inefficient uses of other factors, or of investments. For example, if an airline can no longer pay for fuel, then its airplanes may have to sit idle (e.g., Spanair in February of 2012), which leads to a discontinuous drop in revenue in response to lost new bookings, and so forth. Similarly, if a country or firm’s debt rating is downgraded, that could lead to a discontinuous jump in its cost of capital. Dropping below a critical value might also involve bankruptcy proceedings and additional associated legal costs. This illiquidity then transmits through the network as other organizations either lose asset value or anticipated payments.

\(^2\) We model cross holdings as direct claims on values of organizations for simplicity, but the model extends to all sorts of debt, swaps, or other contracts as discussed in Section \(^8\) in the Supplementary Appendix.
organizations which might also be held by others, and so forth. Diversification refers to how spread out cross holdings are: do many organizations cross hold a particular organization or do just a few? Integration and diversification each exhibit tradeoffs with respect to cascades.

If there is no integration then clearly there cannot be any contagion. As integration increases, the exposure of organizations to each other increases and so contagions become possible. So, on a basic level increasing integration leads to increased exposure which tends to increase the probability and extent of contagions. The countervailing effect here is that an organization’s dependency on its own primitive assets decreases as it becomes integrated. Thus, although integration can increase the likelihood of a cascade once an initial failure occurs, it can also decrease the likelihood of that first failure.

With regards to diversification, there are also tradeoffs but on different dimensions. Here the overall exposure of organizations is held fixed but the set of organizations cross held is varied. With low levels of diversification, organizations can be very sensitive to particular others, but the network of interdependencies is disconnected and overall cascades are limited in depth. As diversification increases, a “sweet spot” is hit where organizations have enough of their cross holdings concentrated in particular other organizations so that a cascade can occur, and yet there are also a rich enough network of cross holdings so that the contagion can be far-reaching. Finally, as diversification is further increased, organizations’ portfolios are sufficiently diversified so that they become insensitive to any particular organization’s failure.

Putting these results together, an economy is most susceptible to wide-spread financial cascades in a middle region, where organizations are partly integrated so that cascades can occur but they still have substantial exposure to idiosyncratic investments that can spark a cascade; and where organizations are partly diversified so that cascades can spread widely but not so diversified so that organizations are immune to each other’s failures. Our results on these tradeoffs include both analytical results on some special cases where the network of cross holdings are tractable, as well as some simulation results on some random cross holding matrices.

We also specialize the model via simulations to examine some specific network structures. One is a network with a clique of large “core” organizations surrounded by many smaller “peripheral” organizations that are each linked to a core organization. This emulates the network of interbank loans. There we see a further nonmonotonicity in integration: if core organizations have low levels of integration then the failure of some peripheral organization is contained with only one core failing, while if core organizations have middle levels of integration then widespread contagions occur, while if core organizations are highly integrated then they become less exposed to any particular peripheral organization and more resistant to peripheral failures. A second model is one with segregation/homophily among sectors. As cross holdings become more sector-specific, particular sectors become more susceptible to cascades, but widespread cascades become less likely. However, the rate at which this happens is nonmonotone in the overall rate of diversification: with lower rates of diversification cascades disappear at lower rates of segregation, as the network begins to fragment.
with lower rates of segregation.

Our next set of results concern a moral hazard problem that increases the economy’s susceptibility to cascades of failures; and is important for understanding policy implications. It might be hoped that organizations will reduce the scope for cascades of failures by minimizing their bankruptcy costs and reducing the threshold values at which they go bankrupt. In fact, the incentives created by financial networks can favor the opposite outcome. We demonstrate this by identifying a novel moral hazard problem – which occurs even absent any government intervention – that incentivizes organizations to increase their bankruptcy costs and their bankruptcy thresholds. Beyond the fact that they may have incentives to undertake overly risky investments - not taking into account the externalities in the network, it can be in organizations’ best interests to increase their bankruptcy costs and to increase the set of circumstances under which they face bankruptcy. This is due to the fact that it helps their bargaining position with respect to receiving aid\(^3\) from other organizations with cross holdings in them.

We also consider what a regulator or government might do to mitigate the possibility of cascades of failures. Preventing a first failure prevents the potential ensuing cascade of failures and it might be hoped that a clever reallocation of cross holdings could achieve this. Unfortunately, we show that any fair exchange of cross holdings or assets involving the organization most at risk of failing makes that organization more likely to fail at some asset prices close to the current asset prices. Making the system unambiguously less susceptible to a first failure necessitates ‘bailing out’ the organization most at risk of failing.

Finally, we illustrate the model in the context of cross holdings of European debt.

While there is a growing literature on networks of interdependencies in financial markets\(^4\), our methodology and results are quite different from any that we are aware of, especially the results on nonmonotonicities in cascades due to integration and diversification.

Independent work by Acemoglu, Ozdaglar and Tahbaz-Salehi (2012b) may be the closest in its motivation of examining comparative statics of how shocks propagate as a function of network architecture, identifying a phase transition via which sufficient interconnections eventually propagate shocks. However, beyond the superficial relationship, the models differ in structure and the results are quite complementary. For instance, they point out that banks do not internalize the externalities in structure and have incentives to form networks that are inefficiently prone to contagion, while we distinguish between diversification and integration and examine different nonmonotonicities due to each and discuss moral hazard problems in bargaining to avoid cascades.

\(^3\) For example, in the form a debt write-down.

2 The Model and Determining Organizations’ Values with Cross Holdings

2.1 Primitive Assets, Organizations, and Cross Holdings

There are \( n \) organizations (e.g., countries, banks, or firms) making up a set \( N = \{1, \ldots, n\} \).

The values of organizations are ultimately based on the values of primitive assets or factors of production – from now on simply assets – \( M = \{1, \ldots, m\} \). For concreteness, a primitive asset may be thought of as a project that generates a net flow of cash over time.\(^5\)

The present value (or market price) of asset \( k \) is denoted \( p_k \). Let \( D_{ik} \geq 0 \) be the share of the value of asset \( k \) held by (that flows directly into) organization \( i \) and let \( D \) denote the matrix whose \((i,k)\)-th entry is equal to \( D_{ik} \). (Analogous notation is used for all matrices in what follows.)

An organization \( i \) can also hold shares of other organizations. For any \( i, j \in N \) the number \( C_{ij} \geq 0 \) is the fraction of organization \( j \) owned by organization \( i \), where \( C_{ii} = 0 \) for each \( i \).\(^6\) It is useful to note that \( C \) can be thought of as a network in which there is a directed link from \( i \) to \( j \) if \( i \) owns a positive share of \( j \) – i.e., if \( C_{ij} > 0 \).\(^7\)

After all these cross-holding shares are accounted for, there remains a share \( \hat{C}_{ii} := 1 - \sum_{j \in N} C_{ji} \) of organization \( i \) not owned by any organization in the system – a share assumed to be positive.\(^8\) This is the part that is owned by outside shareholders of \( i \), external to the system of cross holdings. The off-diagonal entries of the matrix \( \hat{C} \) are defined to be 0.

In terms of interpretations of cross holdings, we have chosen to represent them as shares of value of an organization, so as if they are a sort of equity holding. More generally, cross holdings can involve all sorts of contracts, and so any liability in the form of some payment that is to be paid could be included (fixed debts, swaps, etc.). Directly modeling other sorts of contracting between organizations would complicate the analysis and so we focus on this formulation for now to illustrate the basic issues. Debt and other liabilities and contracts are discussed in Appendix 8 and the approximation by a linear system is illustrated in Figure 9.

\(^5\)The primitive assets could be more general factors: prices of inputs, values of outputs, the quality of organizational know-how, investments in human capital, etc. We model these as abstract investments to keep the exposition simple. For simplicity, we also assume that net positions are nonnegative in all assets.

\(^6\)It is possible to instead allow \( C_{ii} > 0 \), which leads to some straightforward adjustments in the derivations that follow; but one needs to be careful in interpreting what it means for an organization to have cross holdings in itself - which effectively translates into a form of private ownership.

\(^7\)A path from \( i_1 \) to \( i_\ell \) in a matrix \( M \) is a sequence of distinct nodes \( i_1, i_1, \ldots, i_\ell \) such that \( M_{i_r, i_{r+1}} > 0 \) for each \( r \in \{1, 2, \ldots, \ell - 1\} \). A cycle is a sequence of (not necessarily distinct) nodes \( i_1, i_1, \ldots, i_\ell \) such that \( M_{i_r, i_{r+1}} > 0 \) for each \( r \in \{1, 2, \ldots, \ell - 1\} \) and \( M_{i_\ell, i_1} > 0 \).

\(^8\)This assumption ensures that organization’s market values (discussed below) are well-defined. It is slightly stronger than necessary. It would suffice to assume that, for every organization \( i, \) there is some \( j \) such that \( \hat{C}_{jj} > 0 \) and there is a path from \( j \) to \( i \). An organization with \( \hat{C}_{ii} = 0 \) would essentially be a holding company, and the important aspect is to have an economy where there are at least some organizations that are not holding companies.
2.2 Values of Organizations: Accounting and Adjusting for Cross Holdings

In a setting with cross holdings, there are subtleties in determining the “fair market” value of an organization, and the real economic costs of organizations’ failures. Doing the accounting correctly is essential to analyzing cascades of bankruptcy. The basic framework for the accounting was developed by Brioschi, Buzzacchi, and Colombo (1989) and Fedenia, Hodder, and Triantis (1994). In this section, we briefly review the accounting and the key valuation equations in the absence of bankruptcy costs. In ensuing sections, we incorporate bankruptcies and associated discontinuities.

The equity value $V_i$ of an organization $i$ is the total value of its shares – those held by other organizations as well as those held by outside shareholders. This is equal to the value of organization $i$’s primitive assets plus the value of its claims on other organizations:

$$V_i = \sum_k D_{ik} p_k + \sum_j C_{ij} V_j.$$  \hspace{1cm} (1)

Equation (1) can be written in matrix notation as

$$V = Dp + CV$$

and solved to yield

$$V = (I - C)^{-1} Dp.$$  \hspace{1cm} (2)

Adding up equation (1) across organizations (and recalling that each column of $D$ adds up to 1) shows that the sum of the $V_i$ exceeds the total value of primitive assets held by the organizations.\textsuperscript{10} Essentially, each dollar of net primitive assets directly held by organization $i$ contributes a dollar to the equity value of organization $i$, but then is also counted partially on the books of all the organizations that have an equity stake in $i$.\textsuperscript{12}

As argued by both Brioschi, Buzzacchi, and Colombo (1989) and Fedenia, Hodder, and Triantis (1994), the ultimate value of an organization to the economy – what we call the “market” value – is well-captured by the equity value of that organization that is held by its outside investors and creditors. This value captures the flow of real assets that accrues to final investors through that organization. The market value, which we denote by $v_i$, is equal

\textsuperscript{9}Under the assumption that each column of $C$ sums to less than 1, the inverse $(I - C)^{-1}$ is well-defined and nonnegative. (Meyer, 2000, Section 7.10).

\textsuperscript{10}Another way to see that the book values exceed the value of the underlying assets is to note that some of the columns of $(I - C)^{-1}$ sum to more than 1 whenever $C$ is not all zeros.

\textsuperscript{11}This does not directly imply that organizations can arbitrarily increase their equity values simply by acquiring cross holdings in each other. The reason is that such purchases result in debits, i.e. payments for the equity bought, which countervail the inflating force of cross holdings. In the simplest case, when organizations buy equity using cash (a primitive asset), the matrix $D$ changes during the exchange to reflect the decrease in an organization’s cash position.

\textsuperscript{12}This initially counterintuitive feature is discussed in detail by French and Poterba (1991) and Fedenia, Hodder, and Triantis (1994).
to $\hat{C}_{ij}V_i$, and therefore:

$$v = \hat{C}V = \hat{C} (I - C)^{-1} Dp = ADp.$$  \hfill (3)

We refer to $A = \hat{C} (I - C)^{-1}$ as the dependency matrix. It is reminiscent of Leontief’s input-output analysis. Equation 3 shows that value of an organization can be represented as a sum of the value of its claims on primitive assets, with organization $i$ owning a share $A_{ij}$ of $j$’s direct holdings of primitive assets. To see this, suppose each organization fully owns exactly one proprietary asset, so that $m = n$ and $D = I$. In this case, $A_{ij}$ describes the dependence of $i$’s value on $j$’s proprietary assets. It is reassuring that $A$ is column stochastic so that indeed the total values add up to the total values of underlying assets – for all $j \in N$,

$$\sum_{i \in N} A_{ij} = 1.$$

### 2.3 Discontinuities in Values and Bankruptcy Costs

An important part of our model is that organizations can lose productive value in discontinuous ways if their values fall below certain critical thresholds. These discontinuities can lead to cascading failures and also for the presence of multiple equilibria.

For convenience, we refer to these thresholds of discontinuity as bankruptcy thresholds, but they could represent any sort of discontinuity more generally. Indeed, there are many sources of these discontinuities. For example, if an airline can no longer pay for fuel, then it may have to sit its planes idle (e.g, Spanair in February of 2012) which leads to a discontinuous drop in revenue, in response to lost new bookings, and so forth. It might instead be that a country or firm’s debt rating is downgraded which leads to a discontinuous jump in its cost of capital. Dropping below a critical value might also involve bankruptcy proceedings and legal costs. Largely, these discontinuities stem from an illiquidity which then leads to an inefficient use of assets, for example when inputs for assets cannot be purchased and so assets are idle. More generally, given that efficient production can involve a variety of synergies and complementarities, any interruption in the ability to pay for and acquire some factors

\footnote{A way to double check this equation is to derive the market value of an organization from the book value of its underlying assets and cross holdings less the part of its book value promised to other organizations in cross holdings:

$$v_i = \sum_j C_{ij}V_j - \sum_j C_{ji}V_i + \sum_k D_{ik}p_k$$

or

$$v = CV - (I - \hat{C})V + Dp = (C - (I - \hat{C}))V + Dp.$$}

Substituting for the book value $V$ from (2), this becomes

$$v = (C - I + \hat{C})(I - C)^{-1}Dp + Dp = (C - I + \hat{C} + (I - C))(I - C)^{-1}Dp = ADp.$$
of production can lead to discontinuously inefficient uses of other factors, or of investments.

If the value $v_i$ of a organization $i$ falls below some threshold level $v_i$, then $i$ is said to go bankrupt and incurs bankruptcy costs $\beta_i$. \[14\]

It is important to emphasize that bankruptcy costs are based on the (market) values of organizations and not the book values. Bankruptcy occurs when an organization has difficulties in operating, and the artificial inflation in book values that accompanies cross holdings is irrelevant in avoiding a bankruptcy constraint. Cross holdings are important in determining whether a bankruptcy constraint is hit, but because of their affect on the actual values of organizations, not on their book values.

A second comment on these discontinuities is that in many (but not necessarily all) situations a natural cap for $\beta_i$ is $v_i$. That is, to the extent that there is limited liability (in corporations, partnerships, governments, and non-profits), the costs of bankruptcy of organization $i$ fall upon $i$ itself and not its cross holders. Thus, the maximum loss that can be incurred would be the value of organization $i$ itself at the time of bankruptcy, and not more than that, even if it were to owe more. \[15\] \[16\]

A third comment is that the costs could also be allowed to depend on circumstances and so could be made to be a function of various things including prices of assets, values of organizations, cross holdings, and so forth. Again, that would complicate the model, but could be an interesting direction for further research.

Finally, let us say a few words about the relative size of these discontinuities. Recent work has estimated the cost of default to average 21.7 percent of the market value of an organization’s assets, (with substantial variation, see Davydenko, Strebulaev, and Zhao (2012)). Although default costs can be large both absolutely and relative to the value of an organization’s assets (e.g., the size of the recent Greek write-down in debt, or the fire-sale of Lehman Brothers’ assets), it can also be that the effects snowball. Given that a major recession in an economy is only a matter of a change of a few percentage points in its growth rate, when contagions are far-reaching, the particular drops in value of any single organization need not be very large in order to have a large effect on the economy. We develop this observation further in Section \[3.1\] and show in Section \[6\] that a moral hazard problem can result in endogenously high bankruptcy costs.

\[14\] Although $v_i$ is a function of $p$, $v$, $C$, and $D$, as here we usually omit this notation.
\[15\] Also, our bankruptcy costs represent the actual loss in value of the organization that will be incurred independently of $C$, and so has to apply when there are no cross holdings. In that case, the most value that could be lost would be $v_i$.
\[16\] In some cases with discontinuities in values that are present in our model, it might be that bankruptcy is triggered when there is a discontinuous change in the value of some organization due to a contagion of bankruptcy - so because its value of cross holdings in some other organization changes discontinuously. Thus, when entering bankruptcy it could be that $v_i$ is strictly less than $v_i$. Therefore a reasonable extension to the model would be to cap bankruptcy costs at the min $\{v_i, \beta_i\}$, where $v_i$ is the value when the organization is in bankruptcy (properly accounting for all the other bankruptcies that triggered or are triggered by this one, so the equilibrium price). That would not add much insight to our analysis, so we do not impose that here; but we mention it for completeness.
2.4 Including Bankruptcy Costs in Market Values

The valuations in (2) and (3) have easy analogs when we include discontinuities in value due to threshold crossings. The discontinuous drop occurs directly in the value of the organization in terms of additional costs it pays, and so the book value of organization \( i \) becomes:

\[
V_i = \sum_{j \neq i} C_{ij} V_j + \sum_k D_{ik} p_k - \beta_i I_{v_i < v_i}
\]

where \( I_{v_i < v_i} \) is an indicator variable taking value 1 if \( v_i < v_i \) and value 0 otherwise.

This leads to a restatement of (2) as

\[
V = (I - C)^{-1}(Dp - b(v)).
\]

and correspondingly (3) is re-expressed as

\[
v = \hat{C}(I - C)^{-1}(Dp - b(v)) = A(Dp - b(v)).
\]

An entry \( A_{ij} \) of the dependency matrix describes the proportion of \( j \)'s bankruptcy costs that \( i \) pays when \( j \) fails as well as \( i \)'s claims on the primitive assets that \( j \) directly holds. If organization \( j \) goes bankrupt, thereby incurring bankruptcy costs of \( \beta_j \), then \( i \)'s value will decrease by \( A_{ij} \beta_j \).

The solution for organization values in equation (5) encapsulates the network of cross holdings in a clean and powerful form, building on the dependency matrix \( A \).

There can exist multiple solutions to the valuation equation (multiple \( v \)'s satisfying (5)) in the presence of the discontinuities. There always exists a solution, and in fact the set of solutions forms a complete lattice.\(^{17}\)

There are two distinct sources for these multiple equilibria. First, taking other organizations’ values and the values of underlying assets as fixed and given, there can be multiple possible consistent values of organization \( i \) that solve equation (5). There may be a value of \( v_i \) satisfying equation (5) such that \( 1_{v_i \leq v_i} = 0 \) and another value of \( v_i \) satisfying equation (5) such that \( 1_{v_i \leq v_i} = 1 \); even when all other prices and values are held fixed. This source of multiple equilibria corresponds to the standard story of self-fulfilling bank runs (such as those in classic models such as Diamond and Dybvig (1983)). The second source of multiple equilibria is the interdependence of the values of the organizations: the value of \( i \) depends on the value of organization \( j \), while the value of organization \( j \) depends on the value of organization \( i \). There might then be two consistent joint values of \( i \) and \( j \): one consistent value in which both \( i \) and \( j \) fail and another consistent value in which both \( i \) and \( j \) remain solvent. This second source of multiple equilibria is different from the individual bank run concept, as here organizations fail because people expect other organizations to fail, which then becomes self-fulfilling.

\(^{17}\)This holds by a standard application of Tarski’s fixed point theorem, as bankruptcies are complements.
In what follows, we focus on the best case equilibrium in which the minimum number of organizations fail [18]. This allows us to isolate sources of necessary cascades, that are distinct from other self-fulfilling sorts that have already been studied through the sunspot and bank-run literatures.

## 2.5 Measuring Dependencies

The dependency matrix \( A \) appropriately weighs all indirect holdings as well as direct holdings. The central insights of the paper are derived from this \( A \) matrix. In this section we identify some useful properties of the dependency matrix \( A \) and explore its relation to direct cross holdings \( C \).

First, we highlight the different between the dependency matrix \( A \) and the direct cross holdings matrix \( C \). To see how the two might differ, consider the following example of cross holding and outside holdings \( C + \hat{C} \) compared to the associated associated \( A \) matrix. (Recall that \( \hat{C}_{ii} \) is equal to 1 minus the sum of the entries in column \( i \) of \( C \) so that both \( C + \hat{C} \) and \( A \) are column stochastic.)

\[
C = \begin{pmatrix} 0 & 0.75 & 0.75 \\ 0.85 & 0 & 0.10 \\ 0.10 & 0.00 & 0 \end{pmatrix}, \quad A = \hat{C}(I - C)^{-1} = \begin{pmatrix} 0.18 & 0.13 & 0.15 \\ 0.77 & 0.83 & 0.66 \\ 0.05 & 0.04 & 0.19 \end{pmatrix}
\]

The weighted graphs of the above \( C + \hat{C} \) and associated \( A \) are shown in Figure 1, illustrating the substantial differences.

First, note that organization 1 is almost a holding company: it is mostly owned by other organizations, and so the second two entries of row \( A \) are much smaller than those entries in \( C + \hat{C} \).

Also, we see that the outside shareholders of organization 2 have direct and indirect claims to 66% of organization 3’s direct asset holdings, even though the organization has only 10% of the shares of organization 3 directly in cross holdings. Intuitively, as organization 2 directly owns 85% of organization 1, its outside shareholders indirectly have claims to organization 1’s large direct stakes in both organization 2 and organization 3.

Although \( A \) can differ substantially from \( C + \hat{C} \) there are some clear relationships between them, as summarized in the following lemma.

**Lemma 1.** \( \hat{C}_{ii} \) is a lower bound on \( A_{ii} \) but \( A_{ii} \) can be much larger than \( \hat{C}_{ii} \).

1. \( \frac{A_{ii}}{\hat{C}_{ii}} \geq 1 \) for each \( i \), with equality if and only if there are no cycles of cross holdings (i.e. directed cycles in \( C \)) that include \( i \).

---

[18] We show that in this best case equilibrium no organization fails that does not also fail in all other equilibria.
Figure 1: The widths of the edges are proportional to the size of the cross holdings; the arrows point in the direction of the flow of assets: from the organization that is held and to the holder. Outgoing edges in $C + \hat{C}$ reflect the private (final) shareholders’ holdings. The cross holdings and outside holdings measured by $C + \hat{C}$ can be very different from the dependency matrix $A$, measures how each organizations market value depends on the assets held by each organization.

2. For any $n$, there exists a sequence $(C^{(\ell)})$ such that $\frac{A^{(\ell)}}{C^{(\ell)}} \to \infty$ for all $i$.

The magnitude of the lead diagonal terms in $A$ turns out to be critical for determining whether and to what extent failures cascades (section 3.1) and the size of the moral hazard problem we have alluded to (section 5). Lemma 1 demonstrates that the lead diagonal of $A$ can be very similar or much larger than the lead diagonal of $\hat{C}$.

The intuition behind Lemma 1 comes from the fact that cross holdings in one organization that has cross holdings in others, which in turn cross hold others, can lead to very different eventual dependencies on underlying assets than would come from the direct holdings. In particular, an organization that holds others, which then eventually cycle back to own the original, can end up with higher dependency on its own assets than indicated by looking only at its direct asset holdings.

3 Cascades of Failures

In this section we present preliminary results on cascades of failures, showing how failures can be amplified and discussing simple algorithms for identifying hierarchies in cascades.

3.1 Amplification through Cascades of Failures

A relatively small shock to even a small organization can have large effects by triggering a cascade of failures. For simplicity let organization 1 has complete ownership of a single
asset with value \( p_1 \). Consider a small decreases in the value of asset 1 to \( p'_1 < p_1 \) and let \( v_1(p) > v'_1(p) > v_1(p') \) so that 1 fails after this shock. Beyond the loss in value due to the indirect loss of asset value, organizations 2’s value also decreases by \( A_{21}\beta_1 \). If organization 2 also fails, organization 3 absorbs part of two failure costs: \( A_{31}\beta_1 + A_{32}\beta_2 \), and so organization 3 may fail too, and so forth. With each failure the combined shock to the value of each remaining solvent organization increases and organizations that were further and further from failure before the initial shock can get drawn into the cascade. If, for example, the first \( K \) organization end up failing in the cascade, the the cumulative bankruptcy costs to the economy are \( \beta_1 + \cdots + \beta_K \), which can greatly exceed the drop in asset value that precipitated the cascade.

3.2 Who fails in a cascade?

A first step towards understanding how susceptible a system is to a cascade of failures, and how extensive such a cascade might be, is to identify which organizations will fail following a shock. Again, we focus on the best-case equilibrium. Studying the best case equilibrium following a shock identifies the minimum possible set of organizations that will fail. (Results for the worst-case equilibrium are easy analogs identifying the maximum possible set of organizations that will fail.)

3.2.1 Identifying Who Fails When

To understand how and when failures cascade we need to better understand when a fall in asset prices will cause an initial organization to fail and whether the failure of that organization will result in other organizations also failing. Utilizing the dependency matrix \( A \), for each organization \( i \) we can identify the boundary in underlying asset price space below which organization \( i \) must fail, assuming no other organization has failed yet. We can also identify how the failure of one organization affects the failure boundaries of other organizations and so determine when cascades will occur and who will fail in the cascade.

3.2.2 An Example

Suppose there are two organizations, \( i = 1, 2 \), each of whom directly own 100% of a single non-tradeable underlying asset with value \( p_i \) and has a 50% stake in the other. We suppose that organization \( i \) fails when its value falls below 50 and upon failing incurs bankruptcy costs of 50. Thus:

\[
A = \hat{C}(I - C)^{-1} = \left( \begin{array}{cc} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{array} \right) \left[ \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) - \left( \begin{array}{cc} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{array} \right) \right]^{-1} = \left( \begin{array}{cc} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{array} \right)
\]

Organization \( i \) therefore goes bankrupt when \( \frac{2}{3}p_i + \frac{1}{3}p_j < 50 \). Figure 2a shows these failure frontiers for the two organizations. When asset prices are above both failure frontiers,
neither organization fails in the best case equilibrium outcome. One object that we study is
the boundary between this region and the region in which at least one organization fails in
all equilibria. We call this boundary the first failure frontier and it is shown in Figure 2b.

The bankruptcy constraints shown in Figure 2a are not the end of the story. If organiza-
tion \( j \) goes bankrupt, then organization \( i \)'s value falls discontinuously. In effect, through \( i \)'s
cross holding in \( j \) and the reduction in \( j \)'s value, \( i \) bears \( \frac{1}{3} \)rd of \( j \)'s bankruptcy costs of 50.
Organization \( i \) then goes bankrupt if \( \frac{2}{3} p_i + \frac{1}{3} (p_j - 50) < 50 \). We refer to this new bankruptcy
constraint as \( i \)'s failure frontier conditional on \( j \) failing. These conditional failure frontiers
are shown in Figure 2c.

The conditional failure frontiers identify a region of multiple equilibria due to interde-
pendencies in the value of the organizations. As discussed earlier, this is a different source
of multiple equilibria from the familiar bank run story. The multiple equilibria are because
\( i \)'s value decreases discontinuously when \( j \) fails and \( j \)'s value decreases discontinuously when
\( i \) fails. It is then consistent for both \( i \) to and \( j \) to survive, in which case the relevant failure
frontiers are the unconditional ones, and consistent for both \( i \) and \( j \) to fail in which case the
relevant failure frontiers are the conditional ones.

Figure 2d identifies the areas of the failure frontiers where cascades occur in the best case
equilibrium. When asset prices move from being outside the first failure frontier to inside
this region, the failure of one organization precipitates the failure of the other organization;
One organization crosses their unconditional failure frontier and these asset prices are also
inside the other organization’s conditional failure frontier.
3.2.3 A Simple Algorithm for Identifying Cascade Hierarchies

Although all the relevant information about exactly who will fail at what asset prices can be represented in diagrams of the sort in the previous section for simple examples, the number of conditional failure frontiers grows exponentially with organizations and assets and their geometric depiction is infeasible. Thus, while the diagrams provide a useful pedagogical device for introducing ideas, they are of less use practically. In this section, we provide an algorithm that traces how a specific shock that causes one organization to fail propagates. As before, we focus on the best-case equilibrium in terms of having the fewest bankruptcies and the maximum possible $v_i$’s.

At step $t$ of the algorithm, let the set $Z_t$ be the set of bankrupt organizations. Initialize $Z_0 = \emptyset$. At step $t \geq 1$:

Figure 2: With positive cross holdings the discontinuities in values generates by the bankruptcy costs can result in multiple equilibria and cascades of failure.
1. Let $\tilde{b}_{t-1}$ be a vector with element $\tilde{b}_i = \beta_i$ if $i \in Z_{t-1}$ and 0 otherwise.

2. Let $Z_t$ be the set of all $k$ such that row $k$ of the following inequality holds:

$$A \left[Dp - \tilde{b}_{t-1} \right] > v.$$

3. Terminate if $Z_t = Z_{t-1}$. Otherwise return to step 1.

When this algorithm terminates at step $T$ (which it will given the finite number of organizations), the set $Z_T$ corresponds to the set of organizations that fail in the best case equilibrium.

3.2.4 Hierarchies of Bankruptcies

This algorithm provides us with hierarchies of bankruptcies. That is, the various organizations that are added at each step (the new entries in $Z_t$ compared to $Z_{t-1}$) are organizations whose bankruptcy was triggered by the cumulative list of bankruptcy, and in particular they would not have gone bankrupt if not for that accumulation and in particular of those added at the last step. Thus, $Z_1$ represent the first to fail, then $Z_2 \setminus Z_1$ are those whose bankruptcies are triggered by the first to fail and so forth.

Note that the sets depend on $p$ (and $C$ and $D$), and so each situation can result in a different hierarchy. Also, as we shall see below, it is possible to have some $C$ and $D$ such that there are some organizations that are never the first to fail, and others who are sometimes the first to fail and sometimes not.

The hierarchical structure of bankruptcies has immediate and strong policy implications. If any hierarchy can be made empty, then the cascade stops and no further organization will fail. This suggests that one cost effective policy for limiting the effect of bankruptcies should be to target high level hierarchies that consist of relatively few organizations. However, such policies may involve more intervention than is necessary. For example, within a hierarchy there could be a single critical organization, the saving of whom would prevent any further failure in lower hierarchies regardless of whether other organizations in the same cascade failed. Saving an entire hierarchy from failure is sufficient for stopping a cascade, but not necessary. To better inform policy counterfactual scenarios can be run using the algorithm. To determine the marginal effect of saving a set of organizations, the bankruptcy costs of those organizations can be reset to zero and the algorithm run again. This identifies a new set of organizations to fail in the cascade conditional on the intervention. This set of organizations can be compared to the set of organizations that fail under other interventions, including doing nothing.

---

19 The same algorithm can be used to find the set of organizations that fail in the worst case equilibrium by instead initializing the set $Z_0$ to contain all organizations and looking for organizations that will not fail, and so forth.

20 We consider such a policy in Section 5.
It is important to note that the aforementioned exercise must be repeated for any set of underlying asset prices that are of interest. As underlying asset prices change the difference between organizations' values and their bankruptcy constraints change. These changes may be highly correlated depending on the underlying asset holdings. When many organizations have similar exposures to underlying assets they will be relatively close to their failure frontier at the same time. This increases the depth of cascades found by the algorithm.

4 How do Cascades relate to Cross Holdings?

We now turn to our first set of main results. These concern how cascades depend on the level of integration and diversification in cross holdings.

4.1 Integration and Diversification

We say that a financial system becomes more diversified when the number of cross holders in each organization \( i \) weakly increases and the cross holdings of all original cross holder's \( i \) weakly decrease.

Formally, cross holdings \( C' \) are more diversified than cross holdings \( C \) if and only if

- \( C'_{ij} \leq C_{ij} \) for all \( i, j \) such that \( C_{ij} > 0 \), with strict inequality for some \( i, j \) pair, and
- \( C'_{ij} > C_{ij} = 0 \) for some \( i, j \).

Thus, diversification captures the spread in organizations’ cross holdings.

A financial system becomes more integrated if the external shareholders of each organization \( i \) have lower holdings, so that the total cross holdings of the each organization by other organizations weakly increases.

Formally, cross holdings \( C' \) are more integrated than cross holdings \( C \) if and only if

\[
\hat{C}_{ii}' \leq \hat{C}_{ii} \quad \text{for all } i \text{ with strict inequality for some } i.
\]

This is equivalent to the condition that

\[
\sum_{j \neq i} C'_{ij} \geq \sum_{j \neq i} C_{ij},
\]

for all \( i \) with strict inequality for some \( i \).

Thus integration captures the depth or extent of organizations’ cross holdings.

It is possible for a change in cross holdings to both increase diversification and integration. There are changes in cross holdings that increase diversification but not integration and other changes that increase integration but not diversification.

4.2 Essential Ingredients of a Cascade

To best understand the impact of diversification and integration on cascades it is useful to identify three necessary ingredients to have a wide-spread cascade:
I. A First Failure: Some organization must be susceptible enough to shocks in some assets that it fails.

II. Contagion: It must be that some other organization’s are sufficiently sensitive to the first organization’s failure that they also fail.\(^{21}\)

III. Interconnection: It must be that the network of cross holdings is sufficiently connected so that the failures can continue to propagate and are not limited to some small component.

Keeping these different ingredients of cascades in mind will help us disentangle the different effects of changes in cross holdings.

Let us preview of some of the ideas. As we increase integration (say, holding diversification fixed), an organization becomes less sensitive to its own investments but more sensitive to other organizations’ values, and so first failures can become less likely while contagion can become more likely conditional on a failure. This decreases the circumstances that lead to first failures, making things better with respect to I, while it increases the circumstances where there can be contagion, so making things worse with respect to II. Interconnection (III) is not necessarily impacted one way or the other as the network pattern does not change (again, holding diversification fixed). As we increase diversification (say holding integration constant), organizations become less dependent on any particular neighbor, so contagions can be harder to start, but the network becomes more connected and so the extent of a contagion broadens (at least up to a point where the network is fully connected). This decreases the circumstances where there can be contagion, so making things better with respect to II, while increasing the potential reach of a contagion conditional upon one occurring, so making things worse with respect to III.

Understanding this structure makes some things clear. First, integration and diversification affect different ingredients of cascades. Integration affects an organization’s exposure to others compared to its exposure to its own assets; while diversification affects the range of exposure to others and the extent to which an organization is sensitive to any particular other. Second, both integration and diversification ameliorate along some dimension of cascades while leading to a deterioration along some other dimension. The impact of these tradeoffs are nonmonotonic effects on cascades, as we now examine in detail.

4.3 A Specialized Model

To illustrate how increased diversification and increased integration affect the number of organizations that fail in a cascade following the failure of a single organization, we specialize the model.

\(^{21}\)Note that it need not be an immediate cross holder that is the sensitive one. Drops in values propagate through the network, and so the second to fail need not be an immediate cross holder, although that would typically be the case.
Each organization has exactly one proprietary asset so that \( m = n \) and \( D = I \). This keeps the analysis relatively uncluttered, and allows us to focus on the network of cross holdings.

For simplicity, we also start with asset values of \( p_i = 1 \) for all organizations, and have common bankruptcy thresholds \( v_i = \theta v_i \), for a parameter \( \theta \), \( 0 < \theta < 1 \), where \( v_i \) is the starting value of organization \( i \) when all assets are at value 1. In case an organization goes bankrupt it loses its full value, so that \( \beta_i = v_i \) (without going negative if \( v_i < v_i \)).

The cross holdings are derived from an adjacency matrix \( G \), where \( G_{ij} = 1 \) indicates that \( i \) has cross holdings in \( j \) and we set \( G_{ii} = 0 \).

A fraction \( c \) of each organization is held by other organizations, spread evenly among the \( d_i = \sum_j G_{ji} \) organizations that hold it.

Thus, for \( i \neq j \)

\[
C_{ij} = \frac{cG_{ij}}{d_j}.
\] (6)

The remaining \( 1 - c \) of the organization is held by its external shareholders, so that \( \hat{C}_{ii} = 1 - c \). A compact way of writing \( C \) is \( C = cGd^{-1} \).

Holding \( c \) fixed, as \( d_j \) increases more different organizations have cross holdings in \( j \), but those organizations that do have cross holdings in \( j \) have lower cross holdings. Thus, in this model, increasing \( d_j \) in this model increases diversification but not integration.

Holding the underlying graph \( G \) fixed, as \( c \) increases each organization has lower self-holdings but higher cross holdings in the other organizations they already have positive cross holdings in. Thus increasing \( c \) increases integration but not diversification.

### 4.4 Random Networks

To illustrate the affects of increasing diversification and increasing integration on cascades we examine a setting where connections between organizations are formed at random, but with each organization having cross holdings in an expected number of other organizations \( d \).

In particular, we form a directed random graph with link probability \( d/(n-1) \), so that the expected indegree and outdegree of a node is \( d \). In other words, the adjacency matrix of the graph is a (usually not symmetric) matrix \( G \), where \( G_{ij} \) for \( i \neq j \) are i.i.d. Bernoulli random variables which take value 1 with probability \( d/(n-1) \) and 0 otherwise.

To examine the affect of increasing diversification (increasing \( d \)) and increasing integration (decreasing \( c \)), we simulate an organization failing and record the number of organizations that fail in the resulting cascade.

We follow a simple algorithm:

**Step 1.** Generate a directed random network \( G \) on \( n \) nodes with independent link probability \( d/(n-1) \) from \( i \) to \( j \neq i \).
Step 2. Calculate the matrix $C$ from $G$ according to (6), where $\hat{C}_{ii} = .5$.

Step 3. All organizations start with asset values of $p_i = 1$. Calculate organizations' initial values $v_i$ and set $v_i = \theta v_i$ for some $\theta \in (0, 1)$.

Step 4. Pick an organization uniformly at random and drop the randomly picked organization’s own investment value to 0.

Step 5. Assuming all other asset values stay at 1 and only the randomly chosen organization has its asset value that goes to 0, calculate the best equilibrium using the algorithm from Section 3.2.3.

We record the number of failures in the best-case equilibrium.

4.5 The Consequences of Diversification: It Gets Worse Before it Gets Better

For our simulations, we consider $n = 100$ nodes and work with a grid on expected degree $d$ between 1 and 20 (in thirds). We work with values of $\theta \in [0.8, 1]$.

Our first exercise is to vary the level of diversification (the expected degree $d$ in the network) and to see how the number of organizations (out of 100) that fail varies with the diversification.

Figures 3a and 3b illustrate how the proportion of organizations that fail changes as the level of diversification ($d$) is varied (fixing integration at $c = 0.5$). Figure 3a shows the result for a typical level of the bankruptcy threshold ($\theta = 0.93$). We see a nonmonotonicity quite clearly. When $d$ is sufficiently low, 3 or below, then we see the percentage of organizations that fail is limited to below 20. At that level, the network is not connected, and there are various components of organizations that cross hold each other, and any contagion is usually limited to a small component. As $d$ increases (in the range of 5 to 15 other organizations) then we see substantial cascades affecting large percentages of the organizations. In this middle range two things are true: the network is usually completely connected, and firms still hold large enough cross holdings in individual other organizations so that contagion can occur. This is the “sweet spot” where both II and III are maximized - contagion is possible and the cascade is maximized. As we continue to increase diversification the size of cascades is then falling past its peak, as diversification is now lowering the chance that contagion occurs. So, there is constantly a tradeoff between II and III, but initially III dominates as diversification initially leads to dramatic changes in the connectedness of the network, and then II dominates as once the network is connected the main limiting force is
Figure 3: How diversification (the average number of other organizations that an organization cross holds) affects the percentage of organizations failing, averaged over 1000 simulations. The x-axis lists the diversification in terms of the expected degree in the random network of cross holdings.

The extent to which bankruptcy of one organization sparks bankruptcies in others which is decreasing with diversification.

Figure 3b then shows how these effects vary with $\theta$. Higher values of $\theta$ correspond to higher bankruptcy thresholds, and so it becomes easier to trigger contagions. This leads to increases in the curves for all levels of diversification. Essentially, increasing $\theta$ leads to a more fragile economy across the board.

The main results in Subsection 4.7 provide analytical support for the non-monotonicity due to diversification identified in the simulations and helps identify the forces behind the non-monotonicity. With low levels of diversification, contagions may be difficult to start and will frequently die out before affecting many organizations. Condition III is not met, as the network of cross holdings is not connected. Even if all organizations dependent on the failing organization $i$ (those $j$ such that $A_{ij} > 0$) also fail in the cascade, there are sufficiently few organizations affected by the first or subsequent failures that the cascade dies out quickly and is small. As we then increase how spread out cross holdings are across different organizations, we see an increase in the number of organizations that fail in a cascade. The bankruptcy of any one organization infects more other organizations, and more organizations are drawn into the cascade. However, as we continue to diversify cross holdings, eventually the increased diversification leads to a decrease in exposure of any one organization to any other, and so the necessary condition II is not met as no organization depends very much on any other.
4.6 More Firms Fail in More Integrated Systems

Next, we consider the implications of increased integration in our simple model on the depth of cascades, as illustrated in Figure 4.

![Figure 4](image-url)

(a) Five Levels of Integration and the percentage of Organizations Failing as a Function of Expected Degree ($\theta = 0.93, n = 100$)

(b) Five Levels of Integration and the percentage of Organizations Failing as a Function of Expected Degree ($\theta = 0.96, n = 100$)

Figure 4: How integration (the level $c$ of other organizations that an organization cross holds) affects the percentage of organizations failing, averaged over 1000 simulations. The $x$-axis lists the diversification level (the expected degree in the random network of cross holdings). The two figures work with different bankruptcy thresholds and depict how the size of cascades vary with the level of integration $c$ ranging from 0.1 to 0.5.

Figures 4a and 4b illustrate how the proportion of organizations that fail changes as the level of integration is varied from $c = 0.1$ to 0.5, for two different values of $\theta$ (the fraction of initial value that must be retained for an organization to avoid bankruptcy). As integration is increased the curves all shift upward and we see increased cascades.

Although the effects in Figures 4a and 4b show unambiguous increases in cascades as integration increases, they work with levels of $c \leq 0.5$ for which there is not so much of a tradeoff. In particular, for $c \leq 0.5$ the initial firm whose asset price is dropped to 0 always fails (in the range of $\theta \geq 0.8$ considered in the simulations). As $c$ is increased, eventually the integration level begins to help avoid first failures and then we see the tradeoff between I and II that is present in the integration question (diversification is held constant, so III – having to do with the connectedness of the network – is not affected). We can see this in Figure 5.
Figure 5: How integration affects the percentage of “first failures”: the percentage of simulations with at least one organization failing, for various levels of integration $c$ from 0.4 to 0.9, with the $x$-axis tracking diversification (expected degree) in the network. The bankruptcy threshold is constant at $\theta = 0.8$.

Figure 5 shows that as integration increases to very high levels, the percentage of first failures drops: organizations are so integrated that the drop in the value of their own investments is less consequential and so there is no first failure.

To summarize, increasing integration (as long as it is not already very high) makes shocks more likely to propagate to neighbors in the financial network and increases contagion via the mechanism of II. For very high levels of integration, each organization begins to carry something close to the market portfolio, and so a first failure becomes less likely.

4.7 The Consequences of Diversification and Integration: Analytic Results

The randomness in the networks we just considered allows essentially any network to appear with positive, albeit small, probability. This presents a substantial challenge in proving that the non-monotonicity identified by the simulations holds generally. We address by showing that the result can at least be proven for a particular class of networks where degrees are fairly regular.

Let $\mathcal{G}(d, n)$ be the set of all directed graphs with $n$ nodes indexed by an expected degree $d$. In particular, if $d$ is an integer then let the network be regular with all nodes having both in and out degree equal to $d$. If $d$ is not an integer then each node in degree is either $\lfloor d \rfloor$ or $\lceil d \rceil$, and similarly for out degree; and choose the proportions of nodes with each in and out degree such that the overall average in and out degrees are $d$.

A regular random network with degree $d$ is a draw from $\mathcal{G}(d, n)$ uniformly at random.

\footnote{A proportion $d - \lfloor d \rfloor$ have outdegree (indegree) $\lfloor d \rfloor$ and a proportion $\lceil d \rceil - d$ have outdegree (indegree) $\lceil d \rceil$.}
Each organization has a single asset of value 1 (so $D = I$ and $p = (1, \ldots, 1)$). We set all organization’s thresholds $v_i$ to a common $v$, $0 < v < 1$.

For which $d$ will a non-vanishing fraction of organizations fail in expectation following the failure of some asset $i$ picked uniformly at random? Let $E[Z(n, d)]$ be the expected number of organizations that fail following the failure of a randomly selected asset in a random regular network with degree $d$, and $q(d)$ be its limit as $n$ grows:

$$q(d) \equiv \lim_{n \to \infty} \frac{E[Z(n, d)]}{n}.$$

Let $\vec{v}_{\text{min}}$ ($\vec{v}_{\text{max}}$) denote the lowest (highest) initial valuation in the realized network (before any failures and with all assets at value 1).

**Proposition 1.** For the above described initial failure of one asset among organizations in in a regular random network, a non-vanishing fraction of organizations fail if and only if there are intermediate levels of both integration and intermediate diversification:

1. If $c(1-c)\vec{v}_{\text{min}} - v < 1$, then $q(d) = 0$ for all $d$.

2. If $c(1-c)\vec{v}_{\text{max}} - v \geq 1$ then:
   
   (a) for $d < 1$, $q(d) = 0$.
   
   (b) for $d \in \left[1, \left\lfloor \frac{c(1-c)}{\vec{v}_{\text{max}} - v} \right\rfloor \right]$, $q(d) > 0$.
   
   (c) for $d \geq \left\lceil \frac{c}{\vec{v}_{\text{min}} - v} \right\rceil$, $q(d) = 0$.

Proposition 1 reaffirms the non-monotonicity of failures in diversification and integration that we saw in the simulations. The condition that $c(1-c)\vec{v}_{\text{max}} - v \geq 1$ requires intermediate levels of integration, as the left-hand side expression tends to 0 as $c$ tends to 0 or 1. Conditional on that intermediate integration condition being satisfied, the expected number of failures is non-monotonic as we vary the expected degree $d$. Only for intermediate $d$ do a positive proportion of organizations fail following the failure of a single organization.

The intuition for Proposition 1 is relatively straightforward. A standard result in the random graph literature is that there is some threshold level of connectedness at which, in the limit for large graphs, the graph structure suddenly changes from including many small isolated components of vanishing size to containing a giant component of non-vanishing size. In the case of regular random graphs considered in this section that threshold occurs at $d = 1$. Thus for $d < 1$ contagion to positive fraction of organizations is impossible. Moreover, once $d > \frac{c}{1-v}$, a single organization failing will not cause a sufficient decrease in the value of any other organization for another organization to also fail. There can be no contagion. Only for intermediate levels of $d$ can a positive fraction of organizations fail in the cascade.

---

\[23\] Even though nodes all have the same expected degrees and initial asset holdings, the realized network varies (e.g., with respect to in-degrees) and can lead to variations in the $A_{ij}$’s and valuations.
It might seem that the threshold of \( d = 1 \) is low for a giant component in the network to form. This is in part due to the randomness in link formation. In practice networks typically have less random structures and an organization is more likely to have cross holdings in another organization if someone else it has cross holdings in also has cross holding in that organization. In such networks the threshold degree \( d \) for a giant component forming will be higher. We show that in more detail in Subsection 4.8.

Next, we prove a general result about the number of organizations that fail in a given cascade. For this result we relax the assumptions of our simple model and permit any initial cross holdings \( C \). The result is also proved for heterogeneous bankruptcy costs \( \beta \), heterogeneous threshold values \( v \), any direct holdings of assets \( D \) and any underlying asset values \( p \).

Before stating the result we introduce the concept of fair trades. Fair trades are exchanges of cross holdings or underlying assets that leave the (market) values of the organizations’ claims on assets unchanged at current asset prices\(^{24}\). If the assets were freely tradable, these prices would be the unique prices at which any trade occurred\(^{25}\).

**Proposition 2.** For any \( n \) and any initial cross holdings \( C \), conditional upon some initial failure, fair trades that weakly increase \( A_{ij} \) for all \( i \) and \( j \neq i \) also weakly increase the number of organizations that fail in the resulting cascade.

The intuition underlying Proposition 2 is relatively straightforward. As can be seen immediately from equation (5), when organization \( i \) fails and incurs bankruptcy costs \( \beta_i \), it is the \( i \)th column of \( A \) which determines who (indirectly) pays these costs. Increasing \( A_{ij} \) for all \( i \) and \( j \neq i \) increases the share of \( i \)'s bankruptcy costs paid by each other organization. This increases the negative externality \( i \) imposes on each of the other organizations following failure. These other organizations are then more likely to also fail once \( i \) fails and so the number of organizations that fail in the cascade weakly increases.

To tie Proposition 2 back to the affects of increasing integration in our simple model, the following proposition shows how increased integration weakly increases \( A_{ij} \) for all \( i \) and all \( j \neq i \) and strictly increases \( A_{ij} \) for all \( i \) and some \( j \neq i \).

**Proposition 3.** Suppose each organization \( i \) has the same positive proportion \( 0 < c \leq \frac{1}{2} \) of total cross holdings held by other organizations, and these cross holdings are spread evenly among all organizations with positive cross holdings in \( i \), then \( A_{ii} \) is decreasing in \( c \) and \( A_{ij} \) is increasing in \( c \)\(^{26}\).

1. \( \frac{\partial A_{ii}}{\partial c} < 0 \) for each \( i \);

\(^{24}\) So, absent bankruptcy, the values of organizations are the same before and after fair trades.

\(^{25}\) We show in section 6.1 that because of the bankruptcy costs, when direct trades in assets are not possible, trades in cross holdings can occur at different implied asset prices.

\(^{26}\) That is, \( C_{ij} = cG_{ij}/d_j \) for some directed adjacency matrix \( G \) with degree sequence \( d_j = \sum_{i \neq j} G_{ij} \), in which every entry \( d_j \) is at least 1.
2. \( \frac{\partial A_{ij}}{\partial c} \geq 0 \) for all \( i \neq j \);

3. \( \frac{\partial A_{ij}}{\partial c} > 0 \) for all \( i \neq j \) so that there is a path from \( i \) to \( j \) in \( G \).

Note that Proposition 3 does require any assumptions on the underlying graph \( G \) other than each organization being cross held by at least one other.

By Proposition 3 increasing \( c \) weakly increases \( A_{ij} \) for all \( i \) and all \( j \neq i \) so that there is an increase in integration. As long as these trades are fair, which could be ensured by transfers of a numeraire, Proposition 2 can then be applied to show that more organizations will fail in a cascade as the system becomes more integrated.

### 4.8 Alternative Network Structures

We also examined diversification and integration in other random graph models to gain additional insights.

First, in order to capture the structure of the interbank lending market, we examined a core-periphery model where 10 large organizations are completely connected with each other, then a series of 90 smaller organizations are each connected with one of the core organizations. Then ten large core organizations each have assets with initial value 8. The 90 peripheral organizations that are each randomly connected to one of the core organizations each have assets with initial value 1.

We then vary different facets of integration: the level \( C_{CC} \) of cross holdings of each core organization by other core organizations, the level \( C_{PC} \) of cross holdings of each core organization by peripheral organizations, and \( C_{CP} \) of cross holdings of each core organization by other core organizations. The remaining private holdings, \( \hat{C}_{ii} \), of a core organization are \( \hat{C}_{ii} = 1 - C_{CC} - C_{PC} \) and of a peripheral organization are \( \hat{C}_{ii} = 1 - C_{CP} \).

We first explore what happens when a core organization fails. As we see in the left-hand part of Figure 6a, the fraction of peripheral organizations that fail along with the core organization in increasing in \( C_{PC} \). Once the core organizations become sufficiently integrated amongst themselves starting around \( C_{CC} = .29 \) the core organization’s failure begins to cascade to other core organizations, and then wider spread contagion occurs. How far this ultimately spreads is governed by the combination of integration levels.

\footnote{Interestingly, the monotonicity identified in Proposition 3 does not always hold for \( c > 1/2 \). There then exist graph structures where further increases in \( c \) result in the immediate neighbors of \( i \) depending less on \( i \). The increase in \( A_{ij} \) for non-neighbors of \( i \) can come at the expense of both \( A_{ii} \) and \( A_{ij} \) for \( j \) such that \( C_{ij} > 0 \).}

\footnote{As long as initial holdings of this numeraire good are sufficiently large, there will always exist transfers of the numeraire good that will make any given trade fair.}

\footnote{Soromaki et al. (2007) map the US interbank network, identifying a clique of 25 completely connected banks (including the very largest ones), and then thousands of less connected peripheral regional and local banks, based on the Fedpayments system.}

\footnote{Note that in this model the diversification (degree) structure is essentially fixed given the structure of ten completely inter-connected organizations and the peripheral ones each having one connection, except for randomness in which core organization the peripheral organizations are connected to.}
Figure 6: A core-periphery network with 10 core organizations completely connected to each other, and 90 periphery organizations each connected to one core organization (so diversification is set by this structure and fixed). Core assets are eight times more valuable than periphery assets. X-axis is the fraction of each core organization cross held by other core organizations (integration of core to core). Curves correspond to different levels of cross-holdings of each core organization by periphery organizations in Figure 6a and of periphery by core in Figure 6b. Failure threshold is $\theta = .98$.

The more subtle effects are seen in Figure 6b. The curves are layered in terms of integration between the core and periphery $C_{PC}$, with increased integration leading to higher failure rates due to an initial failure of a peripheral organization. However, the magnitude of the failure rates is initially increasing in core integration ($C_{CC} < .25$) and then decreasing in core integration ($C_{CC} > .25$). Initial increases in core-integration enable contagion from one core organization to another, which leads to widespread cascades. Once core integration becomes high enough, however, core organizations become less exposed to their own peripheral organizations, and so then are less prone to fail because of the failure of a peripheral organization.

Second, we considered a model with homophily. In that model, there are ten different groups of ten nodes each. The additional feature is the relative frequency with which nodes connect with others in their own group compared to other groups. This captures the relative integration rate across industries compared to within industries. Varying these relative integration rate leads to the results captured in Figure 7. An obvious effect is that increasing homophily can eventually sever connections between groups of organizations, and so ultimately leads to lower contagion. However, as we see in Figure 7, the curves associated with different diversification (expected degrees $d$) cross each other. With medium diversification
(e.g., \(d = 3\) or \(d = 5\)) there is initially a higher level of contagion than with higher diversification (e.g., \(d = 7\) or \(d = 9\)) since organizations are more susceptible to each other with medium degrees than with high degrees and the network is still connected enough to lead to widespread contagion. However, when homophily is increased the network breaks into separate components at lower levels of homophily when diversification is lower than higher. That is, once at least 95 percent of expected relationships are within own group, then we see lower contagion rates with diversifications \(d = 3, 5\) than with \(d = 7, 9\).

![Figure 7](image)

Figure 7: Ten groups of ten organizations each. Fraction of organizations that fail as a function of the homophily: the fraction of expected cross holdings are to same-type organizations. Curves correspond to different diversification levels (expected degrees \(d\)). Failure threshold is \(\theta = .96\).

Beyond the models, we also examined networks with more extreme degree distributions, such as a power-law distribution. Those results are described in detail in the Supplementary Appendix and are in line with the original regular networks.

5 Avoiding a First Failure

In this section we analyze the changes in the structure of interdependencies between organizations that can help prevent a first failure. We find that for every possible fair trade (as defined in Section 4.6) involving the organization closest to failure the first failure frontier (as defined in section 3.2.2) gets worse somewhere. More specifically, if an organization is close to failing then for any fair trade involving that organization there exists asset values close to the values at which this organization would fail following the fair trade, while it would not have failed before the fair trade.
To state this result formally, it is helpful to introduce some notation. We write firm i’s value assuming no failures at asset prices $p$, cross holdings $C$ and direct holdings $D$ as $v_i(p, C, D)$.

An organization $i$ is closest to failing at asset price $p$, cross holdings $C$, and direct holdings $D$ if there exist a $\lambda > 0$ such that at price $\lambda p$ organization $i$ is in on its failure frontier: $v_i(\lambda p, C, D) = v_i$, while for all other organizations $v_j(\lambda p, C, D) > v_j$.

**Proposition 4.** Fair trades cannot unambiguously help avoid a first failure. At current prices $p$ suppose organization $i$ is closest to failing, with $\lambda$ representing the fraction to which prices can fall before $i$ fails ($v_i(\lambda p, C, D) = v_i$). Consider new cross holdings and direct holdings $C'$ and $D'$ derived from a fair trade that changes $i$’s dependency on the underlying assets. Then, for any $\varepsilon > 0$, there exists a $p'$ within an $\varepsilon$-neighborhood of $\lambda p$ such that $i$ fails at prices $p'$ after the fair trade but not before:

$$v_i(p', C', D') < v_i < v_i(p, C, D).$$

Proposition 4 is more subtle than it appears. It is conceivable that if an organization is away from its frontier, there could exist some (fair) trades that would unambiguously make that organization safer: prone to failure at a smaller set of prices. The proposition shows that there are always tradeoffs: new holdings that avoid failure at some prices will result in failure at some new prices.

## 6 Endogenously High Bankruptcy Costs and Thresholds due to Moral Hazard

Whether an organization fails depends on the size of its bankruptcy threshold, and the impact that its failure has on the rest of the society depends on its bankruptcy costs. If organizations have some control over their bankruptcy costs and thresholds, then we might hope that they would choose to limit these. To the contrary, we show in this section that organizations can have incentives to increase both their bankruptcy costs and thresholds.

### 6.1 Organization Values can be Endogenous

Our previous analysis has typically assumed that exchanges of cross holdings or assets between organizations are fair and at the current market price. That was useful for illustrating the workings of the model and identifying the general effects of organizations being more integrated and have more diversified portfolios. However, the value to an organization of a trade will depend not only on the value of the bundle of assets being received, but also on the implications of the trade for which organizations will fail.

---

$31 ||p' - \lambda p||_1 < \varepsilon$. 

28
For instance, it can be that by relinquishing some holdings (in either assets or another organization) an organization’s value actually increases! This means that we cannot value organizations solely based on their implied underlying asset holdings, but need also to consider the solvency of all other organizations. Trades can be ‘incentive compatible’ when they are not ‘fair’ – both organizations can be willing to engage in unfair trades. We make these points through a simple example.

### 6.2 An example

Consider a world with two assets and two organizations. We begin with a case where asset holdings are $D_1 = (1, 0)$, $D_2 = (0, 1)$. Initial cross holdings are $C_1(0) = (0, 1/3)$ and $C_2(0) = (1/3, 0)$, such that each organization has a one third stake in the other (and so $\hat{C}_{ii} = 2/3$).

From equation (5) it is easily verified that the organizations’ indirect holdings of the underlying assets are

$$A = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$$

With the initial cross holdings organization 1 receives $3/4$ of asset 1’s price while organization 2 receives $1/4$. The opposite is true for the asset 2 price.

Let both asset 1 and asset 2 prices be $p_1 = p_2 = 10$. Thus, without any bankruptcy constraints, the values of the organizations would be $v_1 = v_2 = 10$.

We let $v_1 = 0$ and $v_2 = 11$; Organization 2 goes bankrupt if its asset price is less than 11 while organization 1 never goes bankrupt. Let organization 2’s bankruptcy costs be $\beta_2 = 6$.

This means that if there are no changes in cross holdings, from (5) the values of the two organizations are 8.5 and 5.5. Suppose now that organization 1 can make a transfer to organization 2. If organization 1 were to make a transfer of 1 to organization 2, the values of the two organizations would be 9 and 11. Thus by giving a transfer to organization 2, organization 1 is able to increase its value from 8.5 to 9! Such a transfer might be a direct transfer of cash or implemented through a trade in underlying assets or cross holdings. For example, organization 1 might gift organization 2 an increased stake in itself. Organization 1 is incentivized to ‘save’ organization 2.

Suppose we now extend the above example to permit organization 2 to have some control over his bankruptcy costs $\beta_2$ and bankruptcy threshold $v_2$. For simplicity we suppose that organization 2 can choose from $\beta_2 \in \{2, 6, 10\}$ and from $v_2 \in \{10, 11, 12\}$. Note that

---

32 Values before bankruptcy costs are 10 for organization 1 and 10 for organization 2. Organization 2 therefore fails and its bankruptcy costs of 6 reduces the effective value of asset 2 to 4.

33 One of the lower cost ways in which organization 1 might ‘save’ organization 2 is to simply take over organization 2. This can explain why such buyouts may be observed.

34 All the parameter values in the example can be varied slightly without generating a discontinuous change in the equilibrium. In this sense the example presented is not a knife-edge case.
organization 2 can avoid bankruptcy (at asset prices of 10) absent any trade by choosing $v_2 = 10$. However, such a choice will not be in the best interest of organization 2.

Organization 1 will ‘save’ organization 2 if doing so weakly increases its value. If Organization 2 needs saving ($v_2 > 10$), 1’s value after just saving 2 will be $v_1' = 10 - (v_2 - 10)$ while its value will be $10 - (\beta_2/4)$ if it does not save organization 2. Organization 1 will therefore save organization 2 if and only if $v_2 > 10$ and:

$$\frac{\beta_2}{4} > (v_2 - 10).$$

The left hand side is the increase in value 1 receives from 2 remaining solvent and the right hand side is the cost of saving 2 – the transfer 1 must make to 2 for 2 to remain solvent. Table 1 below shows the transfers that organization 1 will make to organization 2 for different values of $v_2$ and $\beta_2$ that organization 2 can choose. These choices of $v_2$ and $\beta_2$ then result in different values for organization 2 as shown in Table 2:

<table>
<thead>
<tr>
<th>Bankruptcy Costs $\beta_2$</th>
<th>Bankruptcy Costs $\beta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10 8.5 5.5 2.5</td>
</tr>
<tr>
<td>6</td>
<td>11 8.5 11 11</td>
</tr>
<tr>
<td>10</td>
<td>11 8.5 11 11</td>
</tr>
<tr>
<td>Bankruptcy threshold $v_2$</td>
<td>11 0 0 0</td>
</tr>
<tr>
<td>Threshold $v_2$</td>
<td>12 0 0 2</td>
</tr>
<tr>
<td>12</td>
<td>13 0 0 0</td>
</tr>
<tr>
<td>13</td>
<td>13 8.5 5.5 2.5</td>
</tr>
</tbody>
</table>

Table 1: Transfer made from 1 to 2

Table 2: Value of 2

As can be seen in tables 1 and 2, for a fixed bankruptcy threshold organization 2 is only saved when his bankruptcy costs are sufficiently large. Conditional on being saved 2’s value is increasing in his bankruptcy threshold and conditional on not being saved, organization 2’s value is decreasing in his bankruptcy threshold. For sufficiently high bankruptcy thresholds organization 2 is never saved and for sufficiently low bankruptcy threshold organization 2 doesn’t fail. To maximize its value, organization 2 must set the highest bankruptcy costs they can and then carefully choose their bankruptcy threshold so that organization 1 is just incentivized to save it. In this example, this requires organization 2 choosing a bankruptcy threshold of 12 and bankruptcy costs of 10.

Of course, if organizations can commit not to bail each other out, then these moral hazard problems can be avoided. However, firms have a fiduciary obligation to maximize shareholder value, even if this involves bailing out a failing organization they have a stake in. This can make it difficult for organization to commit not to bail out another, and absent such a commitment device organizations can have strong incentives to increase their bankruptcy costs and manipulate their bankruptcy thresholds, possibly increasing or decreasing them.

The moral hazard problem in this example occurs absent any market intervention by the government. Bankruptcy costs alone are sufficient for moral hazard problems to arise. However, the moral hazard problem also distorts organizations’ investment decisions, both in terms of their investments into risky projects and their investments in cross holdings.
The moral hazard problem arises because organizations do not fully bear their bankruptcy costs. It is because other organizations pay (indirectly through the devaluation in $i$) some of organization $i$’s bankruptcy costs ($\beta_i$), and these other organizations are thus willing to expend resources bailing out $i$. As the proportion of $i$’s bankruptcy costs that $i$ pays is given by $A_{ii}$, a natural measure of the moral hazard problem is $1 - A_{ii}$. When $1 - A_{ii} = 0$ there is no moral hazard problem and the extent of the moral hazard problem is monotonic in $1 - A_{ii} = 0$ in the following sense: If $1 - A_{ii}$ increases, such that some shares of $i$ redistributed from outside shareholders of $i$ to other organizations, then any organization that previously would have bailed out $i$ faces weakly stronger incentives to bailout $i$, while organizations who previously would not have found it profitable to bailout $i$ may now find it profitable to bailout $i$.

We saw in Section 3.1 that cascades of failure can occur amplifying and propagating shocks if bankruptcy costs are sufficiently large and bankruptcy thresholds are sufficiently high. The analysis in this section has identified an endogenous mechanism through which organizations are willing to invest in increasing their bankruptcy costs and possibly their bankruptcy thresholds. Although such investments are valuable to an organization only in the event that they are bailed out, in an uncertain world such bailouts may or may not be forthcoming the misalignment of incentives due to the moral hazard can result in system endogenously conducive to cascades of failure.

7 Illustration of European Debt Cross Holdings

We close the paper with an illustration of the model with data on the cross holdings of debt among six European countries (France, Germany, Greece, Italy, Portugal and Spain). We include this as a proof of concept, and emphasize that the crude estimates that we use for cross holdings make this noisy enough that we do not see the conclusions as robust, but merely as illustrative of the methodology. Moreover, in the simulations, when a country declares bankruptcy it defaults on 50% of its obligations to foreign countries. Such losses may corresponds more closely to a sequence of disorderly bankruptcies than the more orderly writing down of Greek debt that has occurred over time. For the purposes of this illustrative exercise, we treat these countries as a closed system with no holdings by other countries outside of these six.

7.1 The Data

Data on the cross holdings are for the end of December 2011 from the BIS (Bank for International Settlements) Quarterly Review (Table 9B). The data used for this exercise are the consolidated foreign claims of banks from one country on another country. The data looks at the immediate borrower rather than the final borrower when a bank from a country
different from the final borrower serves as an intermediary.\footnote{For illustrative purposes, we examine holdings at a country level, so that all holdings of Italian debt by banks or other investors in France are treated as the entity “France.” It would be more accurate to disaggregate and build a network of all organizations and investors, if such data were available.}

This gives following \textit{raw} cross holdings matrix where the \textit{column} represents the country whose debt is being held and the row is the country which holds that debt. So, for example, through their banking sectors Italy owes France $329,550 Million while France only owes Italy $40,311 Million.

\[
\begin{pmatrix}
(\text{France}) & (\text{Germany}) & (\text{Greece}) & (\text{Italy}) & (\text{Portugal}) & (\text{Spain}) \\
(\text{France}) & 0 & 174862 & 1960 & 40311 & 6679 & 27015 \\
(\text{Germany}) & 198304 & 0 & 2663 & 227813 & 2271 & 54178 \\
(\text{Greece}) & 39458 & 32977 & 0 & 2302 & 8077 & 1001 \\
(\text{Italy}) & 329550 & 133954 & 444 & 0 & 2108 & 29938 \\
(\text{Portugal}) & 21817 & 30208 & 51 & 3188 & 0 & 78005 \\
(\text{Spain}) & 115162 & 146096 & 292 & 26939 & 21620 & 0 \\
\end{pmatrix}
\]

To convert the above matrix into our fractional cross holdings matrix, $C$, we then estimate the total amount of debt of each country by using an estimate of the ratio of foreign to domestic holdings of 1/3 in line with estimates of by Reinhart and Rogoff (2011). Then, since $A = \hat{C}(I - C)^{-1}$ leads to it follows that

\[
A = \begin{pmatrix}
(\text{France}) & (\text{Germany}) & (\text{Greece}) & (\text{Italy}) & (\text{Portugal}) & (\text{Spain}) \\
(\text{France}) & 0.71 & 0.13 & 0.13 & 0.17 & 0.07 & 0.11 \\
(\text{Germany}) & 0.18 & 0.72 & 0.12 & 0.11 & 0.09 & 0.14 \\
(\text{Greece}) & 0.00 & 0.00 & 0.67 & 0.00 & 0.00 & 0.00 \\
(\text{Italy}) & 0.07 & 0.12 & 0.03 & 0.70 & 0.03 & 0.05 \\
(\text{Portugal}) & 0.01 & 0.00 & 0.02 & 0.00 & 0.67 & 0.02 \\
(\text{Spain}) & 0.03 & 0.03 & 0.02 & 0.02 & 0.14 & 0.68 \\
\end{pmatrix}
\]

The $A$ matrix can be pictured as a weighted directed graph, as in Figure. The arrows show the way in which decreases in value flow from country to country. For example, the arrow from Greece to France represents the value of France’s claims on Greek assets. The area of the ovals represent the value of each country’s direct holdings of primitive assets. All dependencies of less than 5% have been excluded from the Figure (but appear in the table above).

We treat the investments in primitive assets as if each country holds its own fiscal stream which is used to pay for the debt and presume that the values of these fiscal streams are proportional to GDP. So, $D = I$ and $p$ is proportional to the vector of countries’ GDPs. Normalizing Portugal’s GDP to 1, the initial values in 2011 are:

[p. 32]
Figure 8: Interdependencies in Europe: The $A$ dependency matrix. The widths of the arrows are proportional to the sizes of the cross holdings; the area of the oval for each country is proportional to its underlying asset values.

\[
v_0 = Ap = \begin{pmatrix} 0.71 & 0.13 & 0.13 & 0.17 & 0.07 & 0.11 \\ 0.18 & 0.72 & 0.12 & 0.11 & 0.09 & 0.14 \\ 0.00 & 0.00 & 0.67 & 0.00 & 0.00 & 0.00 \\ 0.07 & 0.12 & 0.03 & 0.70 & 0.03 & 0.05 \\ 0.01 & 0.00 & 0.02 & 0.00 & 0.67 & 0.02 \\ 0.03 & 0.03 & 0.02 & 0.02 & 0.14 & 0.68 \end{pmatrix} \cdot \begin{pmatrix} 11.6 \\ 14.9 \\ 1.3 \\ 9.2 \\ 1.0 \\ 6.3 \end{pmatrix} = \begin{pmatrix} 12.7 \text{ (France)} \\ 14.9 \text{ (Germany)} \\ 0.8 \text{ (Greece)} \\ 9.4 \text{ (Italy)} \\ 0.9 \text{ (Portugal)} \\ 7.1 \text{ (Spain)} \end{pmatrix}.
\]

7.2 Cascades

To illustrate the methodology, we consider a simple scenario. The failure thresholds are set to $\theta$ times 2008 values (where those values are calculated in the same way as the values above, being proportional to 2008 GDP values instead of 2011 and again normalized by Portugal’s 2011 value set to 1). If a country fails, then the loss in value is $v_i/2$, so that it devalues by
We examine the best equilibrium values for various levels of $\theta$. Greece’s value has already fallen by well more than ten percent, and so it has hit its failure point for all of the values of $\theta$. We then raise $\theta$ to various values and see which cascades occur.

<table>
<thead>
<tr>
<th>Value of $\theta$</th>
<th>.9</th>
<th>.93</th>
<th>.935</th>
<th>.94</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Failure</td>
<td>Greece</td>
<td>Greece</td>
<td>Greece</td>
<td>Greece, Portugal</td>
</tr>
<tr>
<td>Second Failure</td>
<td></td>
<td>Portugal</td>
<td>Spain</td>
<td></td>
</tr>
<tr>
<td>Third Failure</td>
<td></td>
<td>Spain</td>
<td>France</td>
<td></td>
</tr>
<tr>
<td>Fourth Failure</td>
<td></td>
<td>France, Germany</td>
<td>Germany, Italy</td>
<td></td>
</tr>
<tr>
<td>Fifth Failure</td>
<td></td>
<td>Italy</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Hierarchies of Cascades in the Best Equilibrium Algorithm, as a Function of the Failure Threshold $\theta$.

We see that Portugal is the first failure to be triggered by a contagion. Although it is not so exposed to Greek debt directly, the fact that its GDP has dropped substantially means that it is triggered once we get to $\theta = .935$. Once Portugal fails, then Spain fails due to its poor initial value and its exposure to Portugal. Then the large size of Spain, and the exposure of France and Germany to Spain cause them to fail. Pushing $\theta$ up to .94 causes Portugal to fail directly, and then leads to a similar sequence (and increasing $\theta$ further would not change the ordering, just would cause failures at earlier levels. Interestingly, Italy is the last in each case: this is due to its low exposure to others’ debts. Its GDP is not particularly strong, but it does not hold much of the other countries.

Clearly the above exercise is based on rough numbers and ad hoc estimates for the default thresholds and a closed (six country) world. Nonetheless, it illustrates the simplicity of the approach and makes it clear that much more accurate simulations could be run with access to precise cross holdings data, default costs and thresholds.

We re-emphasize that the cascades are (hopefully!) off-the-equilibrium-path, but that understanding the dependency matrix and the hierarchy of the cascades can help improve policy intervention and also help predict bailout structure.

8 Concluding Remarks

Based on a simple model of cross holdings and discontinuities in values, we have examined cascades in financial networks. We have highlighted several important features: First, diversification and integration are usefully distinguished as they have different effects on financial contagions. Second, both diversification and integration face tradeoffs of competing effects, changing the exposures that organizations have to each other and to their own assets. These tradeoffs result in nonmonotonic effects where middle ranges are the most dangerous with respect to cascades of failures. The specifics of the tradeoffs have some intuitive relationships to the structure of the network, such as its core-periphery and segregation structure.
Finally, potential cascades introduce interesting moral hazard problems, where organizations can have incentives to increase their bankruptcy thresholds and or costs in order to receive larger bailouts at an interim stage when they are close to bankruptcy.

A fully endogenous study of the network of cross holdings and of asset holdings is a natural next step. The moral hazard issues that we have demonstrated suggest that modeling the endogenous structures will be delicate and that a simple general equilibrium approach will not suffice. This presents interesting challenges for future research.

References


Appendix: Proofs

Proof of Lemma 1

One representation of $A$ is as the following infinite sum, known as the Neumann series:

$$A = \hat{C} \sum_{p=0}^{\infty} C^p$$  \hspace{1cm} (7)

Parts (i) and (iii) follow immediately from this representation of $A$.

Part (ii) can be proved by considering $C$ such that $\hat{C}_{ii} = \epsilon$ for every $i$ and $C_{ij} = (1 - \epsilon)/(n - 1)$ and taking $\epsilon \to 0$. Then $A$ tends to the matrix with all entries $1/n$.

Proof of Lemma 2

Recall that

$$A = \hat{C}(I - C)^{-1},$$

or alternatively that

$$A = \hat{C} \sum_{t=0}^{\infty} C^t.$$
Let $\overline{C}$ be the matrix for which we set $\overline{C}_{ij} = \frac{C_{ij}}{1 - C_{jj}}$.

Then,

$$A \leq \overline{C} \sum_{t=0}^{\infty} \overline{C}_t^t.$$

Note that $\overline{C}$ is a column stochastic matrix. It follows that $\overline{C}^{t-1}$ is also a column stochastic for any $t \geq 1$ (because it is a column-stochastic matrix raised to a power). Write $\overline{C}_t = \overline{C} \overline{C}^{t-1}$.

From this, given the fact that $\overline{C}^{t-1}$ is column stochastic for each $t$, it follows that the $ij$-th entry of $\overline{C}_t$ is no more than $\max_{k \neq i} \max_{k \neq j} \frac{C_{ik}}{1 - C_{kk}}$. Also, note that for $t = 0$, the $ij$-th entry of $\overline{C}^t$ when $j \neq i$ is 0. Thus, for $i \neq j$,

$$A_{ij} \leq \overline{C}_{ii} \sum_{t=0}^{\infty} \overline{C}_t^t \max_{k \neq i} \overline{C}_{ik}.$$ 

Then since $1/\sum_{t=1}^{\infty} \overline{C}_t = \overline{c}/(1 - \overline{c})$ it follows that

$$A_{ij} \leq \overline{C}_{ii} \frac{\overline{c}}{1 - \overline{c}} \max_{k \neq i} \overline{C}_{ik},$$

This is the claimed expression for $j \neq i$. For $j = i$ we also have the $ii$-then entry of $\overline{C}^0$ being 1. The simplifications for $\overline{C}_{ii} = 1 - c$ for all $i$ follow directly.

**Proof of Proposition 1**: A contagion of failures flows through organizations’ indegrees. When organization $i$ fails organization $j$ might also fail if $C_{ji} > 0$. We therefore wish to know when there is a giant component in indegree such that following only organizations’ indegrees a non-vanishing fraction of organizations can be reached in expectation from an organization selected uniformly at random. From results from Newman, Strogatz and Watts (?), for a regular random graph with indegree and outdegrees in either $\lfloor d \rfloor, \lceil d \rceil$, a giant component (in indegree) appears for:

$$\sum_{k \in \{\lfloor d \rfloor, \lceil d \rceil \}} \sum_{k' \in \{\lfloor d \rfloor, \lceil d \rceil \}} (2kk' - k - k')\theta_{kk'} \geq 0.$$ 

Simplifying the above equation there will be a giant component for $d \geq 1$ and so for all $d < 1$ it follows that $q(d) = 0$.

Suppose that organization $j$ has holdings in organization $i$ and recall that if organization $i$ fails, organization $j$’s value will decrease by $A_{ji}$. A lower bound on $A_{ji}$ for a regular random graph can be found by considering a tree network. If contagion would occur within a tree then any feedback effects can only increase contagion. We therefore have that $A_{ji} \geq \frac{c(1-c)}{|d|}$. Organization $j$ will therefore fail, following the failure of organization $i$ if:

$$\overline{c}_{max} - \frac{c(1-c)}{|d|} < \frac{c}{|d|}$$
We therefore get contagion for sure within a component for any \( |d| < \frac{c(1-c)}{v_{\text{max}}-\bar{v}} \). Combining
the above results we therefore have that \( q(d) > 0 \) for \( d \in \left(1, \left\lfloor \frac{c(1-c)}{v_{\text{max}}-\bar{v}} \right\rfloor \right) \).

Finally, we derive an upper bound on possible contagion. From Lemma \( \text{2} \) \( A_{ji} \leq \frac{c}{|d|} \) for each \( i \neq j \). It follows that there will be no contagion if:

\[
\bar{v}_{\text{min}} - \frac{c}{|d|} > \bar{v} \quad |d| > \frac{c}{\bar{v}_{\text{min}} - \bar{v}}
\]

There is thus no contagion for \( d > \left\lfloor \frac{c}{\bar{v}_{\text{min}} - \bar{v}} \right\rfloor \). This completes the proof of Proposition \( ?? \).

Proof of Proposition \( \text{2} \):

Following the failures of organizations \( Z_{k-1} \), the value of organization \( i \) is:

\[
v_i(Z_{k-1}) = \sum_{j \notin Z_{k-1}} a_{ij} D_{jk} p_k + \sum_{j \in Z_{k-1}} a_{ij} (D_{jk} p_k - \beta_j)
= v_i(\emptyset) - \sum_{j \in Z_{k-1}} a_{ij} \beta_j.
\]

As fair trades hold constant \( v_i(\emptyset) \), this equation shows that the value of organization \( i \) given bankruptcies \( Z_{k-1} \) is weakly decreasing in \( A_{ij} \) for all \( j \neq i \). Holding fixed the hierarchies in which all other organizations fail, after a weak increase in \( A_{ij} \) for all \( i \) and all \( j \neq i \), if organization \( i \) failed in hierarchy \( k \) it will now fail (weakly) sooner in hierarchy \( k' \leq k \) and if organization \( i \) did not fail in any hierarchy it might now fail in some hierarchy.

Moreover, as bankruptcies are complementary, if organization \( i \) fails strictly sooner in hierarchy \( k' \) weakly more organizations will be included in all subsequent failure sets \( Z_{k''} \), for all \( k'' > k' \). This is because more bankruptcy costs are summed over in the above equation when calculating a firm’s value in each bankruptcy hierarchy.

Proof of Proposition \( \text{3} \):

Let \( C = Gd^{-1} \) and note that by \( (??) \) and the Neumann series, we may write

\[
A = (1 - c) \sum_{t=0}^{\infty} c^t C^t,
\]

so

\[
\frac{\partial A}{\partial c} = (1 - c) \sum_{t=1}^{\infty} t c^{t-1} C^t - \sum_{t=0}^{\infty} c^t C^t
\]

38
\[ -I + \sum_{t=1}^{\infty} (t(1 - c) - c)c^{t-1}\mathbf{C}^t. \]

Since \( c \leq \frac{1}{2} \), every term in the summation over \( t \) is nonnegative. Moreover, \( c^{t-1}\mathbf{C}^t \) has a strictly positive entry whenever there is a path of length \( t \) from \( i \) to \( j \) in \( \mathbf{C} \), or equivalently in \( \mathbf{G} \). This shows claims 2 and 3 in the proposition. To verify claim 1, note that every column of \( \mathbf{A} \) sums to 1. Claim 3 along with the assumption that every node in \( \mathbf{G} \) has at least one neighbor shows that every column has an off-diagonal entry that strictly increases in \( c \); and no off-diagonal entry decreases by claim 2. So the diagonal entry must strictly decrease in \( c \).

**Proof of Proposition 4.**

As any trade involving organization \( i \) must change composition of \( i \)'s dependency on underlying assets, after any trade there must exists a price vector \( \mathbf{p}'' \) within an \( \epsilon \) neighborhood of \( \lambda \mathbf{p} \), such that \( v_i(\mathbf{p}'', \mathbf{C}', \mathbf{D}'|Z = \emptyset) \neq v_i(\mathbf{p}'', \mathbf{C}, \mathbf{D}|Z = \emptyset) = \underline{v}_i \). For the Proposition to be false, it must then be that \( v_i(\mathbf{p}'', \mathbf{C}', \mathbf{D}'|Z = \emptyset) > v_i(\mathbf{p}'', \mathbf{C}, \mathbf{D}|Z = \emptyset) \). Define price \( \mathbf{p}' \) such that \( \frac{1}{2}\mathbf{p}'' + \frac{1}{2}\mathbf{p}' = \lambda \mathbf{p} \). As \( ||\mathbf{p}' - \lambda \mathbf{p}||_1 = ||\mathbf{p}'' - \lambda \mathbf{p}||_1 \) and \( \mathbf{p}'' \) is within an \( \epsilon \) neighborhood of \( \lambda \mathbf{p} \), \( \mathbf{p}' \) is also within an \( \epsilon \) neighborhood of \( \lambda \mathbf{p} \).

By the linearity of firms value, absent any bankruptcy, and as the trade was fair

\[
\frac{1}{2}v_i(\mathbf{p}'', \mathbf{C}', \mathbf{D}'|Z = \emptyset) + \frac{1}{2}v_i(\mathbf{p}', \mathbf{C}', \mathbf{D}'|Z = \emptyset) = v_i(\lambda \mathbf{p}, \mathbf{C}', \mathbf{D}'|Z = \emptyset) = \underline{v}_i = v_i(\lambda \mathbf{p}, \mathbf{C}', \mathbf{D}'|Z = \emptyset) = v_i(\lambda \mathbf{p}, \mathbf{C}, \mathbf{D}|Z = \emptyset) = \frac{1}{2}v_i(\mathbf{p}'', \mathbf{C}, \mathbf{D}|Z = \emptyset) + \frac{1}{2}v_i(\mathbf{p}', \mathbf{C}, \mathbf{D}|Z = \emptyset).
\]

Thus as \( v_i(\mathbf{p}'', \mathbf{C}', \mathbf{D}'|Z = \emptyset) > v_i(\mathbf{p}'', \mathbf{C}, \mathbf{D}|Z = \emptyset) \),

\[ v_i(\mathbf{p}', \mathbf{C}', \mathbf{D}'|Z = \emptyset) < \underline{v}_i < v_i(\mathbf{p}', \mathbf{C}, \mathbf{D}|Z = \emptyset). \]
Supplementary Appendix: Not for Publication

Bounds on the Dependency Matrix

We provide some useful upper bounds on the possible values of the dependency matrix $A$. Let $\bar{c} = \max_k 1 - \hat{C}_{kk}$, and

$$A_{ij} = \hat{C}_{ii} \frac{\bar{c}}{1 - \bar{c}} \max_{k \neq i} \frac{C_{ik}}{1 - \hat{C}_{kk}}$$

and

$$A_{ii} = \hat{C}_{ii} \left(1 + \frac{\bar{c}}{1 - \bar{c}} \max_{k \neq i} \frac{C_{ik}}{1 - \hat{C}_{kk}} \right).$$

**Lemma 2.** $A_{ij}$ is an upper bound on $A_{ij}$ for all $i$ and $j$. Therefore, if $\hat{C}_{ii} = 1 - c$ for all $i$, so that each organization holds $c$ of its holdings in other organizations and $1 - c$ in itself, then $A_{ij} \leq \max_{k \neq i} C_{ik}$ for each $i$ and $j \neq i$, and $A_{ii} \leq (1 - c) + \max_{k \neq i} C_{ik}$.

Multiple Equilibria and Discontinuities in Organizations’ Values

In the absence of any bankruptcy issues, equation (5) is a standard pricing equation describing how the values of organizations depend on the primitive asset values $v = A[Dp]$. The novel and interesting part of equation (5) comes from the bankruptcy terms $b(v)$. These bankruptcy terms generate several complexities that equation (5) illuminates.

In particular, the presence of bankruptcy introduces several forms of discontinuity which result in multiple equilibria and cascades of default. Discontinuities in the value of a given organization $i$ can come from two sources. The basic form is that the bankruptcy costs of organization $i$ can be triggered when the values of other organizations or underlying assets fall which then lead $i$ to hit its bankruptcy constraint. The other form is due to another organization, in which $i$ has cross holdings, hitting its bankruptcy constraint which then leads to a discontinuous drop in the value of $i$’s holdings and consequently its value.

In terms of multiplicities of equilibria, there are also different ways in which these can occur. The first is that taking other organizations’ values and the value of underlying assets as fixed and given, there can be multiple possible consistent values of organization $i$ that solve equation (5). There may be a value of $v_i$ satisfying equation (5) such that $1_{v_i \leq \mathbb{E}_i} = 0$ and another value of $v_i$ satisfying equation (5) such that $1_{v_i \leq \mathbb{E}_i} = 1$; even when all other prices and values are held fixed. This generates the a first source of multiple equilibria corresponding to the standard story of self-fulfilling bank runs (such as those in classic models such as Diamond and Dybvig (1983)).

The second is the interdependence of the values of the organizations: the value of $i$ depends on the value of organization $j$, while the value of organization $j$ depends on the value of organization $i$, and given the discontinuities possible in prices due to bankruptcy costs, there can be multiple solutions. There might then be two consistent joint values of
\( i \) and \( j \): one consistent value in which both \( i \) and \( j \) fail and another consistent value in which both \( i \) and \( j \) remain solvent. This second source of multiple equilibria is different from the individual bank run concept, as here organizations fail because people expect other organizations to fail, which then becomes self-fulfilling.

Although governments may be able to give assurances such as insuring deposits that manipulate expectations regarding the self-fulfilling value of a single organizations, it seems more difficult to control expectations when an organization’s value depend on the expected values of many other organizations. For example, an organization’s value can depend on the expected value of an organization that falls under the regulatory oversight of another government. Suppose organizations \( A \) and \( B \) have cross holdings in each other and organization \( B \) also has cross holdings in organization \( C \). Investors in organization \( A \) may then become less confident investors will keep their money in organization \( B \), or less confident the investors in \( B \) have confidence in them or in the investors in \( C \), and so on.

**Details: Cascades of Default in Europe**

Here we provide the calculations of the \( v_i \)'s. These are based on the peak GDPs from 2008. The normalized GDPs (relative to Portugal’s GDP in 2011) are:

\[
\begin{pmatrix}
12.0 \\
15.3 \\
1.5 \\
9.7 \\
1.1 \\
6.7
\end{pmatrix}
\]

This leads to values based on the \( A \) matrix of:

\[
v_0 = Ap = \begin{pmatrix}
0.71 & 0.13 & 0.17 & 0.07 & 0.11 \\
0.18 & 0.72 & 0.12 & 0.11 & 0.09 & 0.14 \\
0.00 & 0.00 & 0.67 & 0.00 & 0.00 & 0.00 \\
0.07 & 0.12 & 0.03 & 0.70 & 0.03 & 0.05 \\
0.01 & 0.00 & 0.02 & 0.00 & 0.67 & 0.02 \\
0.03 & 0.03 & 0.02 & 0.02 & 0.14 & 0.68
\end{pmatrix}
\begin{pmatrix}
12.0 \\
15.3 \\
1.5 \\
9.7 \\
1.1 \\
6.7
\end{pmatrix} =
\begin{pmatrix}
13.1 \text{ (France)} \\
15.4 \text{ (Germany)} \\
1.0 \text{ (Greece)} \\
9.8 \text{ (Italy)} \\
1.0 \text{ (Portugal)} \\
7.5 \text{ (Spain)}
\end{pmatrix}
\]
Thus
\[
\begin{pmatrix}
13.1 \text{ (France)} \\
15.4 \text{ (Germany)} \\
1.0 \text{ (Greece)} \\
9.8 \text{ (Italy)} \\
1.0 \text{ (Portugal)} \\
7.5 \text{ (Spain)}
\end{pmatrix}, \quad \text{and} \quad \beta = \frac{\theta}{2}
\begin{pmatrix}
13.1 \text{ (France)} \\
15.4 \text{ (Germany)} \\
1.0 \text{ (Greece)} \\
9.8 \text{ (Italy)} \\
1.0 \text{ (Portugal)} \\
7.5 \text{ (Spain)}
\end{pmatrix}.
\]

**Debt and other Liabilities**

The model is easily adapted to general sorts of cross liabilities beyond the linear cross holdings. These could reflect any sort of debt or other contractual agreement, which could be contingent on the market value of the organizations (for instance, the debt cannot exceed the organization’s market value if there is limited liability). If we let \( L_{ji}(V) \) be the amount owed to \( j \) by \( i \) as a function of book value, and \( L \) the corresponding matrix (with 0’s on its diagonal, as an organization cannot have debt to itself) book values become:

\[
V_i = \sum_{j \neq i} C_{ij}V_j + \sum_{j \neq i} (L_{ij}(V) - L_{ji}(V)) + \sum_k D_{ik}p_k - \beta_iI_{v_i < v_i}.
\]

This leads to book values of

\[
V = (I - C)^{-1}(Dp - (L(V) - L^T(V))1 - b(v)). \tag{8}
\]

where \( T \) indicates transpose, and correspondingly market values are then

\[
v = \hat{C}(I - C)^{-1}(Dp - (L(V) - L^T(V))1 - b(v)). \tag{9}
\]
Using the Dependency Matrix

This section validates the direct use and manipulation of the dependency matrix $A$. Proposition 5 shows that absent any discontinuities (i.e. with bankruptcy costs of zero for all organizations), any change in $C$ or $A$ can be represented as changes in $D$ alone. Proposition 6 then identifies a simple necessary and sufficient condition for the $A$ to be valid – that is for there to exist direct cross holdings $C$ it can be derived from. This second result allows one to directly manipulate $A$. 

Figure 9: Approximation of a debt model by a linear model.
**Proposition 5.** Assuming there are no bankruptcies, for any \( D, C \) there is a \( D', C' \) with \( C' \) being the identity that results in the same organization values for any underlying asset prices \( p \). Similarly, for any \( A, D \) there exists \( D' \) with \( \hat{C} \) being the identity that results in the same organization values for any underlying asset prices \( p \).

Proposition 5 follows directly from letting

\[
D' = (\hat{C}(I - C)^{-1})D = AD.
\]

Thus, in the absence of bankruptcy, it is simply the indirect holdings of underlying assets that matter, and so one can equivalently work with them in understanding organizations’ values.

The proposition implies that instead of considering trades in cross holdings, when we are working to understand what might trigger a first bankruptcy (so that none have yet occurred) there is always some trade in underlying assets that replicates the trade in cross holdings.

However, in practice, at least some of the underlying assets are non-tradeable and so can only be held through cross holdings. To work in the underlying asset space we therefore want to know when trades of underlying assets can be replicated through an exchange of cross holdings, keeping the organizations’ asset holdings \( D \) constant. Proposition 6 provides necessary and sufficient conditions on \( A \) for it to be a valid representation of some \( C \).

**Proposition 6.** There exists a valid cross holdings matrix \( \hat{C} + C \) (i.e. one that is column stochastic, contains non-negative entries and has strictly positive entries on the lead diagonal) that generates \( A \) if and only if \( A^{-1} > 0 \) for all \( i \) and \( A^{-1} \leq 0 \) for all \( i \) and all \( j \neq i \).

**Proof of Proposition 6:** Recall from (5) that

\[
A = \hat{C}(I - C)^{-1}.
\]

If \( A \) is invertible, manipulating this equations yields that:

\[
\begin{align*}
A^{-1} &= (\hat{C}(I - C)^{-1})^{-1} \\
A^{-1} &= (I - C)\hat{C}^{-1} \\
A^{-1}\hat{C} &= I - C \\
C &= I - A^{-1}\hat{C}
\end{align*}
\]

If we can represent the right hand side of this equation just in terms of the \( A \) matrix, we will have found a way to map an \( A \) matrix into a \( C \) matrix. We will then just need to

\[36\]If all underlying assets were freely tradeable then there would be no reason for any cross holdings. Any portfolio of claims to underlying assets held through cross holdings could be replicated as direct holdings and without any risk of devaluation through bankruptcy.
find conditions under which the $C$ matrix we are deriving is column stochastic and has all non-negative elements (and strictly positive elements on the lead diagonal) when added to $\hat{C}$. When these conditions are met, the $A$ matrix will have an associated valid $C$ matrix it can be derived from and we can work directly with it.

Considering entry $(i, i)$ of this matrix equation, and recalling that $\hat{C}$ is a diagonal matrix:

$$C_{ii} = 1 - (A^{-1})_{ii}\hat{C}_{ii}.$$  

Since $C_{ii} = 0$ by assumption, we find $\hat{C}_{ii} = 1/(A^{-1})_{ii}$. This puts the left hand side of (10) in terms of just $A$. Letting $\hat{C}$ be the matrix thus defined, set

$$S = I - A^{-1}\hat{C}. \quad (11)$$

Thus the matrix $A$ can be derived from a valid $C$ (equal to the $S$ matrix in equation (11)) if and only if (i) $S + \hat{C}$ is column stochastic such that column $j$ of $S$ sums to $1 - \hat{C}_{jj}$ and (ii) all entries of $S + \hat{C}$ are non-negative and the lead diagonal is strictly positive.

First we prove that $S + \hat{C}$ is column stochastic. All valid $A$ matrices are column stochastic and so $A^{-1}$ is also column stochastic. To see this let $1$ be the vector of ones such that $1A = 1$. This is the definition of $A$ being column stochastic. Now post multiply by $A^{-1}$. We then find that $1 = 1A^{-1}$ and so $A^{-1}$ is also column stochastic.

As $A^{-1}$ is column stochastic, $\sum_{i=1}^{n}(A^{-1})_{ij}\hat{C}_{jj} = \hat{C}_{jj} \sum_{i=1}^{n}(A^{-1})_{ij} = \hat{C}_{jj}$. Adding $\hat{C}$ to both sides of equation (11) we then have that:

$$\sum_{i=1}^{n} S_{ij} + \hat{C}_{ij} = \sum_{i=1}^{n} I_{ij} - (A^{-1})_{ij}\hat{C}_{jj} + \hat{C}_{ij} = 1 - \hat{C}_{jj} + \hat{C}_{jj} = 1$$

As $S$ is always column stochastic, there exists a valid $C$ representation of $A$ if and only if all entries of $S$ are non-negative.

From equation (11) the elements of $S$ are:

$$S_{ii} + \hat{C}_{ii} = 1 - \frac{(A^{-1})_{ii}}{(A^{-1})_{ii}} + \frac{1}{(A^{-1})_{ii}} = \frac{1}{(A^{-1})_{ii}} \quad \text{and} \quad S_{ij} + \hat{C}_{ij} = \frac{-A^{-1}_{ij}}{A_{jj}},$$

for all $i$ and all $j \neq i$. Thus all elements of $S$ are well-defined and weakly positive if and only if $(A^{-1})_{ii} > 0$ and $(A^{-1})_{ij} \leq 0$ for all $i$ and all $j \neq i$.

**Proof of Proposition ??**: If no organization fails, then their market values are:

$$v = ADp.$$ 

In order to be best case safest, we need to maximize the percentage loss that any organization can suffer without going bankrupt. As all assets have positive value, this requires
equalizing the proportional loss in value each organization must suffer to go bankrupt. If this was not equalized, reallocating assets at the margin from the set of organizations furthest from their bankruptcy constraints to those organizations closest to them would increase the percentage loss in value that any organization can suffer without going bankrupt. Thus, in a best case safest asset allocation:

\[ \mathbf{v} = \mathbf{A} \mathbf{D} \mathbf{p} = \theta \mathbf{v} \]

for some scalar \( \theta \).

As by assumption it is possible for all organizations to fail at the same time and so:

\[ \sum_i \sum_k D_{ik} p_k - \sum_i \beta_i < \sum_i v_i \]

As bankruptcy costs are a constant proportion of the value of organizations’ direct asset holdings and as \( \mathbf{A} \) is column stochastic:

\[ \sum_j \sum_i \sum_k (1 - \gamma) A_{ij} D_{ik} p_k < \sum_i v_i \]

Using equation 8:

\[ (1 - \gamma) \theta \sum_i v_i < \sum_i v_i \]

and so \( (1 - \gamma) \theta < 1 \).

Suppose now all organizations fail. In this case:

\[ \mathbf{v} = \mathbf{A}(\mathbf{D} \mathbf{p} - \mathbf{\beta}) = \mathbf{A} \mathbf{D} \mathbf{p} (1 - \gamma) = (1 - \gamma) \theta \mathbf{v} < \mathbf{v} \]

Thus, in the worst case equilibrium, all organizations fail.